Lévy based Cox processes for spatio-temporal modelling

Michaela Prokešová[1], Gunnar Hellmund[1], Eva Vedel Jensen[1]

[1] Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark

Abstract: We define the class of Lévy based Cox processes. Such processes have a random intensity function that can be expressed in terms of an integral of a weight function with respect to a Lévy basis. For the spatio-temporal Lévy driven Cox processes the range of integration is determined by so-called ambit set which defines the dependency on the past. We derive first and second order characteristics of the processes and discuss the stationary case in more detail.

Keywords: Lévy bases; Cox processes; Spatio-temporal point processes.

1 Introduction

Let Υ be a Borel subset of $\mathbf{R}^n$ and λ a random locally integrable function on Υ. A point process X on Υ is called a Cox process if conditionally on λ (so called driving intensity) it is a Poisson point process with intensity function λ. We will consider models for spatio-temporal Cox processes with driving intensity defined by means of an integral of a weight function with respect to a Lévy basis L on Υ

$$\rho(\xi) = \int_{\Upsilon} k(\eta, \xi)\, L(\,\mathrm{d}\eta) \qquad \xi \in \Upsilon.$$

The Lévy driven Cox processes (LCP) are defined directly by $\lambda(\eta) = \rho(\eta)$, the log Lévy driven Cox processes (LLCP) are defined by $\lambda(\eta) = \exp(\rho(\eta))$.

To be able to obtain specific spatio-temporal dependence structure we employ the concept of ambit processes previously successfully used for modelling of turbulence and growth (i.e. Barndorff Nielsen and Schmiegel (2006)). For proofs and further details of the results on LCP and LLCP we refer to Gunnar et.al. (2006).

2 Ingredients of the model

The Lévy basis $(L(A), A \in \mathcal{B}(\Upsilon))$ is an independently scattered random measure characterized by its cumulant function $C(v \ddagger L(A)) = \mathbf{E}\left(\mathrm{e}^{i\,v\,L(A)}\right)$

$$C(v \ddagger L(A)) = iv\,a(A) - \frac{1}{2}\,v^2\,b(A) + \int_{\mathbb{R}} (\,\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\,U(\,\mathrm{d}r, A),$$

with a a signed measure, b a measure, $U(\mathrm{d}r, A)$ a Lévy measure for fixed A and a measure on $\mathcal{B}(\Upsilon)$ for fixed $\mathrm{d}r$.
Without loss of generality (Rajput and Rosinski (1989)) we assume existence of a measure μ on Υ such that U factorizes as $U(\mathrm{d}r, \mathrm{d}\eta) = V(\mathrm{d}r, \eta)\mu(\mathrm{d}\eta)$ with $V(\mathrm{d}r, \eta)$ being a Lévy measure for fixed η, and a, b being absolutely continuous with respect to μ with densities $\tilde{a}(\eta)$, $\tilde{b}(\eta)$, respectively. Considering now the random variable with the cumulant function

$$C(v \ddagger L'(\eta)) = iv\tilde{a}(\eta) - \frac{1}{2}v^2\tilde{b}(\eta) + \int_{\mathbb{R}} (\mathrm{e}^{ivr} - 1 - ivr\mathbf{1}_{[-1,1]}(r))\, V(\mathrm{d}r, \eta),$$

we get a disintegrated representation of the Lévy basis

$$C(v \ddagger L(\mathrm{d}\eta)) = C(v \ddagger L'(\eta))\, \mu(\mathrm{d}\eta).$$

Let $\mathcal{S} \subseteq \mathbb{R}^d$ be a d-dimensional Borel set and L be a Lévy basis on $(\Upsilon, \mathcal{B}(\Upsilon) = (\mathcal{S} \times \mathbb{R}, \mathcal{B}(\mathcal{S} \times \mathbb{R}))$. Let each point $(x, t) \in \Upsilon$ be associated with an ambit set $A_t(x) \subset \mathcal{S} \times (-\infty, t]$ which defines the causal correlation cone. The spatio-temporal process $\{\rho(x, t) : (x, t) \in \Upsilon\}$ (the ambit process) is then for each point (x, t) defined as

$$\rho(x, t) = \int_{A_t(x)} f_{(x,t)}(y, s) L(\mathrm{d}(y, s)).$$

Thus the ambit set $A_t(x)$ determines the part of L that influences the behaviour of ρ in the point (x, t). We assume all the ambit sets and weight functions being uniformly bounded and define the spatio-temporal LCP and LLCP by such ρ.

3 Results on spatio-temporal LCP

The LCP driven by $\rho(x, t)$ is stationary for $\mathcal{S} = \mathbb{R}^d$ if $A_t(x) = \{(y, s) : (y - x, s - t) \in A_0(0)\}$ for all (x, t), $f_{(x,t)}(y, s) = f(y - x, s - t)$ for all (x, t), (y, s) and the Lévy basis is homogeneous (i.e. distribution of $L'(y, s)$ does not depend on (y, s) and μ is proportional to the Lebesgue measure).

The non-negativity of $\lambda = \rho$ for LCP implies non-negativity of the used Lévy basis L and thus LCP are generalization of the class of shot noise Cox processes studied in Møller (2003).

The intensity function $\Lambda = \mathbf{E}\,\lambda$ of a LCP is given by

$$\Lambda(x, t) = \mathbf{E}\,\rho(x, t) = \int_{A_t(x)} f_{(x,t)}(y, s)\ \mathbf{E}\,(L'((y, s)))\, \mu\,(\mathrm{d}(y, s)),$$

and the pair correlation function $g((x_1, t_1), (x_2, t_2))$ of LCP is given by

$$\begin{aligned} &g((x_1, t_1), (x_2, t_2)) \\ &= 1 + \frac{\int_{A_{t_1}(x_1) \cap A_{t_2}(x_2)} f_{(x_1,t_1)}(y, s) f_{(x_2,t_2)}((y, s))\, \mathrm{Var}\,(L'((y, s)))\, \mu\,(\mathrm{d}(y, s))}{\Lambda(x_1, t_1)\, \Lambda(x_2, t_2)}. \end{aligned}$$

Here $\mathbf{E}\,L'((y,s))$ and $\mathrm{Var}L'((y,s))$ denote the mean value and the variance of the random variable L' from the disintegrated representation of the Lévy basis.

For the stationary case, if $\mathbf{E}\,L'$ and $\mathrm{Var}L'$ exist they do not depend on (y,s). Moreover μ is a multiple of the Lebesgue measure and we can write $\mu(\,\mathrm{d}(y,s)) = K\,\mathrm{d}(y,s)$ for some $K > 0$ Thus the formulas simplify to

$$\begin{aligned} \Lambda(x,t) &= \Lambda = K\,I\,\mathbf{E}\,L', \\ g((x_1,t_1),(x_2,t_2)) &= 1 + \frac{\mathrm{Var}\,L'}{(\mathbf{E}\,L')^2}\,\frac{1}{K}\,\frac{I(x_2-x_1,t_2-t_1)}{I^2}, \end{aligned}$$

where

$$I = \int_{A_0(0)} f_{(0,0)}(y,s)\,\mathrm{d}(y,s),$$

and

$$\begin{aligned} &I(x_2-x_1,t_2-t_1) \\ &= \int_{A_0(0)\cap A_{t_2-t_1}(x_2-x_1)} f_{(0,0)}(y,s) f_{(0,0)}(y-(x_2-x_1), s-(t_2-t_1))\,\mathrm{d}(y,s). \end{aligned}$$

Thus for such stationary Lévy driven Cox process we can see nicely how the effects due to the disintegrated parts of the Lévy basis and the integrals of the weight function on the ambit sets combine in the pair correlation function.

3.1 Cumulative process for LCP

Let X be a LCP. We define the cumulative point process X^C by

$$X^C(B) = X(B \times [T_0, T_1]) \qquad B \in \mathcal{B}(\mathcal{S}),\ T_0 < T_1 \in \mathbf{R}.$$

Because of the linear structure of the Lévy driven Cox process model the cumulative process is also a Lévy driven Cox process and its driving intensity has the form

$$\lambda^C(x) = \int_{T_0}^{T_1} \int_{A_t(x)} f_{(x,t)}(y,s) L(\,\mathrm{d}(y,s))\,\mathrm{d}t = \int_{\Upsilon} f_x^C(y,s)\,L(\,\mathrm{d}(y,s)),$$

with a new weight function

$$f_x^C(y,s) = \int_{T_0}^{T_1} \mathbf{1}((y,s) \in A_t(x))\, f((x,t),(y,s))\,\mathrm{d}t$$

defined for each $x \in \mathcal{S}$. Consequently we can obtain formulas for the intensity and pair correlation function of the cumulative process X^C in the same way as we did for the original process X.

4 Results on spatio-temporal LLCP

The LLCP is stationary under the same conditions as the LCP.
Let us denote the kumulant function of a random variable Y by $K(v \ddagger Y) = C(-iv \ddagger Y)$ and the n-th order product densities of a point process by $m^{(n)}$.

If the n-th order product density of a LLCP X on Υ exists, then

$$\begin{aligned} m^{(n)}((x_1,t_1),\dots,(x_n,t_n)) &= \mathbf{E}\prod_{i=1}^{n}\lambda(x_i,t_i) \\ &= \exp\left(\int_\Upsilon K\Big(\sum_{i=1}^{n}\mathbf{1}_{A_{t_i}(x_i)}(y,s) f_{(x_i,t_i)}(y,s) \ddagger L'((y,s))\Big)\mu(\,\mathrm{d}(y,s))\right), \end{aligned}$$

which implies the following results.

The intensity of a LLCP is given by

$$\Lambda(x,t) = \exp\left(\int_\Upsilon K(\mathbf{1}_{A_t(x)}(y,s)\, f_{(x,t)}(y,s) \ddagger L'((y,s)))\,\mu(\,\mathrm{d}(y,s))\right),$$

and pair-correlation function $g((x_1,t_1),(x_2,t_2))$ of a LLCP is equal to

$$\begin{aligned} \exp\Bigg(\int_{A_{t_1}(x_1)\cap A_{t_2}(x_2)} \Big[& K\big(f_{(x_1,t_1)}(y,s) + f_{(x_2,t_2)}(y,s) \ddagger L'((y,s))\big) \\ & - K\big(f_{(x_1,t_1)}(y,s) \ddagger L'((y,s))\big) - K\big(f_{(x_2,t_2)}(y,s) \ddagger L'((y,s))\big)\Big]\,\mu(\,\mathrm{d}(y,s))\Bigg). \end{aligned}$$

Acknowledgments: M.P. acknowledges support of the Carlsberg foundation.

References

Barndorff-Nielsen, O.E., and Schmiegel, J. (2006) Ambit processes; with applications to turbulence and tumour growth. To appear in Proceedings of the Abel Symposium 2005, Springer.

Brix, A. (1999) Generalized gamma measures and shot-noise Cox processes. *Adv. Appl. Prob.*, **31**, 929–953.

Hellmund, G., Prokešová, M., Jensen, E.B.V. (2006) Spatio-temporal Lévy based models for point processes. (in preparation).

Møller, J. (2003). Shot noise Cox processes. *Adv. Appl. Prob.*, **35**, 614–640.

Rajput, B.S., and Rosinski, J. (1989). Spectral Representation of Divisible Processes *Probability Theory and Related Fields*, **82**, 451–487.