

An asymptotic series approach to qualocation methods

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Abstract

In this paper we develop an asymptotic analysis of qualocation methods when applied to a particular class of pseudodifferential equations. The main result is the existence of an error expansion between the numerical solution and an optimal projection over the splines. As by-products of this expansion we obtain some estimations of pointwise convergence and an asymptotic expansion between the exact and the numerical solution under the action of regularising operators. In addition to this, using the error expansion we deduce sufficient conditions to obtain qualocation methods of higher order for some particular equations. Finally, we give a numerical experiment which corroborates the theoretical results.

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1 Introduction

This paper is devoted to a full asymptotic analysis of qualocation methods for a certain class of periodic pseudodifferential equations. The class of equations (associated to what we will call expandable operators) we will be working on includes boundary integral equations [3] on smooth closed curves of the plane. In fact, it can be proved that the set of operators in our equations is that of periodic classical pseudodifferential of integer order [11, 13, 27].

Qualocation methods (qualocation is a compression of quadrature-modified collocation) form a recent family of numerical methods for the approximation of pseudodifferential equations. This family consists of three different groups of methods. The first group

is that of semidiscretizations of spline–spline Petrov–Galerkin schemes via substitution of the test integration process by a suitable quadrature rule [8, 24]. The second one includes quadrature approximation of outer integrals in a spline–trigonometric Petrov–Galerkin method [20, 21]. Finally, the so-called “tolerant” qualocation only discretizes the test integral in the bilinear form but not in the right hand side (cf. [26]).

Although these methods can be seen as a further step towards full discretization of Petrov–Galerkin methods, both their analysis and properties (and also the look of the discrete equations) are closer to those of collocation methods. In fact, collocation methods are enclosed in this wider set. The main point for this proximity is that an apparent additional discretization of a Petrov–Galerkin method can give better approximation properties than the original scheme. The class of qualocation methods has been steadily developing in the nineties with the study of quadrature rules valid for more general equations. However, stability (especially when the operator is neither strongly nor oddly elliptic) is an open question in many cases.

The present paper contributes to the error analysis of qualocation methods by adopting a different approach. We basically expand the error as an asymptotic series in powers of the discretization parameter h (i.e., a possibly divergent series where remainders of partial sums can be bounded by higher order terms). In addition to the inherent interest these asymptotic series may have (including justification of Richardson extrapolation for convergence acceleration and a posteriori error estimation), this approach gives several by-products. On the one hand, we are able to find sets of superconvergence points in strong norms. On the other hand, by choosing adequate rules that cancel coefficients in the series we show how convergence can be improved. Moreover, the existence of a nonvanishing term proves optimality of the error bounds.

The mathematical tools used for this analysis include a wide set of Fourier techniques. The numerical solution is not compared with the exact one but with an optimal order projection onto spline spaces D_h^d whose properties had already been studied [2, 12, 13]. This has the advantage of concentrating all the error terms due to the trial space in one simple operator and pushing the first term due to the whole numerical scheme to the weakest norm convergence order. The projection operator D_h^d is a matching of central Fourier coefficients. The consistency error $A(D_h^d u - u)$ is then studied and afterwards tests are applied to develop a full asymptotic series of the tested error. This is done by using error expansions of compound quadrature rules applied to the L^2 product of splines by smooth functions and supposes a sort of nonstandard Euler–Maclaurin formula. Once stability is formulated as an inf-sup condition (via identification of the couple of test spline and quadrature rule with the set of functionals it defines), expansions are easily converted into asymptotic series of the error in several norms. These results established, we apply the series to show how terms can be eliminated in particular situations. A final section is devoted to a single numerical example corroborating the theoretical results. In this example adequate numerical quadrature has been applied to preserve the good properties of the method.

Since the method includes collocation schemes, this paper generalizes results in [6] applied to logarithmic integral equations of the first kind with variable coefficients. In comparison with this work, novelties are plenty, not only in the much wider scope of equations and methods but also in the different mathematical techniques. A previous paper

[17] had also treated asymptotic expansions of collocation methods for pseudodifferential equations. The comparison there with the exact solution (instead of going through the projection D_h^d) imposed the error analysis under the action of a regularising functional, which limited the strength of the results and did not covene to a further study of numerical integration.

2 The functional setting and the equation

In this section we introduce the periodic Sobolev spaces as well as the set of periodic pseudodifferential operators and equations, which constitute the frame where we will be working in the sequel. In the space of 1-periodic infinitely often differentiable complex valued functions of a real variable

$$\mathcal{D} := \left\{ \varphi : \mathbb{R} \rightarrow \mathbb{C} \mid \varphi \in \mathcal{C}^\infty(\mathbb{R}), \quad \varphi = \varphi(1 + \cdot) \right\},$$

we consider the metric

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|\varphi_1^{(k)} - \varphi_2^{(k)}\|_\infty}{1 + \|\varphi_1^{(k)} - \varphi_2^{(k)}\|_\infty}$$

($\|\cdot\|_\infty$ denotes the maximum norm). There is a well-known identification of 1-periodic complex valued distributions on the real line with the elements of \mathcal{D}' , the dual space of \mathcal{D} . For instance, $f \in L^1(0, 1)$ defines an element of \mathcal{D}' through

$$\mathcal{D} \ni \varphi \longmapsto \langle f, \varphi \rangle := \int_0^1 f(t)\varphi(t)dt.$$

The corresponding periodic distribution is simply the 1-periodization of f to the whole of \mathbb{R} . Therefore, we will refer to $T \in \mathcal{D}'$ as a 1-periodic distribution. For these we can calculate the distributional Fourier coefficients

$$\widehat{T}(m) := \langle T, \phi_{-m} \rangle, \quad m \in \mathbb{Z},$$

where

$$\phi_m(x) := \exp(2\pi i m x).$$

Obviously, distributional and classical Fourier coefficients of $f \in L^1(0, 1)$ coincide.

We can define the periodic Sobolev spaces (see [15] Chapter 8, [29])

$$H^s := \left\{ u \in \mathcal{D}' \mid \sum_{m \neq 0} |m|^{2s} |\widehat{u}(m)|^2 < \infty \right\}, \quad s \in \mathbb{R}$$

endowed with the norms

$$\|u\|_s := \left[|\widehat{u}(0)|^2 + \sum_{m \neq 0} |m|^{2s} |\widehat{u}(m)|^2 \right]^{1/2}.$$

Then for all s , H^s is a Hilbert space. If $s > t$, $H^s \subset H^t$ with dense and compact injection. Clearly $H^0 = L^2(0, 1)$ and the norms coincide by Parseval's identity.

The inner product in H^0

$$(u, v)_0 := \widehat{u}(0)\overline{\widehat{v}(0)} + \sum_{m \neq 0} \widehat{u}(m)\overline{\widehat{v}(m)} = \int_0^1 u(t)\overline{v(t)} dt$$

can be extended to give a reciprocal representation of the duality between H^s and H^{-s} for all $s \in \mathbb{R}$, with the equality

$$\|u\|_{-s} = \sup_{0 \neq v \in H^s} \frac{|(u, v)_0|}{\|v\|_s}.$$

A linear map $A : \mathcal{D}' \rightarrow \mathcal{D}'$ such that $A : H^s \rightarrow H^{s-n}$ is bounded for all $s \in \mathbb{R}$ with a given $n \in \mathbb{Z}$ is called a pseudodifferential operator (ψ do in short) of order (at most) n . The differential and inverse-differential operators

$$D^j u := \sum_{m \neq 0} (2m\pi i)^j \widehat{u}(m) \phi_m, \quad j \in \mathbb{Z} \quad (1)$$

are ψ do of order j , satisfying $D^j D^k = D^{j+k}$ for all j, k , $D^0 u = u - \widehat{u}(0)$. In addition, D^1 is the differentiation operator in \mathcal{D}' . The periodic Hilbert transform

$$Hu := \sum_{m \neq 0} \text{sign}(m) \widehat{u}(m) \phi_m = \sum_{m > 0} \widehat{u}(m) \phi_m - \sum_{m < 0} \widehat{u}(m) \phi_m$$

is a ψ do of order zero, which admits a singular integral representation when restricted to H^0 (see [29]) and satisfies $H^2 = D^0$. The operators D^j and HD^j will be taken now as monomials of the algebra of expandable operators.

We say that A is an expandable pseudodifferential operator of order n and we write $A \in \mathcal{E}(n)$, if there exist two sequences of functions $(a_j)_{j=-\infty}^n, (b_j)_{j=-\infty}^n \subset \mathcal{D}$ such that for all M integer

$$A = \sum_{j=-M}^n a_j D^j + \sum_{j=-M}^n b_j HD^j + K_M \quad (2)$$

with K_M a ψ do of order $-M - 1$. For $A \in \mathcal{E}(n)$ we denote its expansion by

$$A \stackrel{\text{exp}}{=} \sum_{j=-\infty}^n a_j D^j + \sum_{j=-\infty}^n b_j HD^j.$$

On the other hand, the first term of the expansion $a_n D^n + b_n HD^n$ is called the principal part of A . Expandable ψ do's can be characterized as classical periodic pseudodifferential operators of integer order (cf. [27]).

Following classical denominations (see [24]) we say that A is strongly elliptic if there exists $\varphi \in \mathcal{D}$ such that

$$\text{Re}[\varphi(a_n + b_n)] > 0, \quad \text{Re}[(-1)^n \varphi(a_n - b_n)] > 0.$$

If HA is strongly elliptic, we say that A is oddly elliptic. Let us remark that these concepts are not exclusive: an operator can be strongly and oddly elliptic at the same time.

If $a \in \mathcal{D}$ we equally denote by a to the order zero ψ do of multiplication by a . Then

$$aD^n - D^n a \in \mathcal{E}(n-1). \quad (3)$$

In case $n \geq 1$, this is just the Leibnitz rule, whereas the cases $n \leq 0$ are equally simple. Also

$$aH - Ha \in \mathcal{E}(-\infty), \quad (4)$$

which means that this operator is a ψ do of arbitrarily low degree and as such admits a trivial expansion ($a_j = b_j = 0$ for all j).

The main purpose of this work is the study of qualocation methods for the approximate solution of the equation

$$Au = f, \quad (5)$$

where $A \in \mathcal{E}(n)$. These equations include a wide range of boundary integral operators such as integro-differential equations of Cauchy type and logarithmic integral equations of the first and second kind [3], as can be seen in [13] Appendix.

3 A class of qualocation methods

Qualocation methods include two different families of discrete methods to solve (5). On the one hand we have semidiscretizations of the exterior integral of a spline-spline Petrov-Galerkin method by means of compound quadrature rule as in [8, 24]. On the other hand we have the same kind of semidiscretization of Petrov-Galerkin methods with splines as trial functions and trigonometric polynomials as tests [20, 21]. We will concentrate on the first and leave the second for a final section.

We begin by introducing the discrete spaces. Let N be a positive integer, $h := 1/N$ and $x_i := ih$ for all $i \in \mathbb{Z}$. The space of 1-periodic smoothest splines of degree $d \geq 1$ over the grid $\{x_i\}_{i \in \mathbb{Z}}$ is defined by

$$S_h^d := \left\{ u \in C^{d-1} \mid u|_{[x_i, x_{i+1}]} \in P_d \right\}$$

where P_d is the set of polynomials of degree at most d . The space of periodic piecewise constant functions will be consequently denoted S_h^0 . It is well known that $S_h^d \subset H^s$ for all $s < d + 1/2$.

Consider now a basic quadrature rule

$$Lu := \sum_{j=1}^J \omega_j u(\xi_j) \approx \int_0^1 u(t) dt,$$

where $0 \leq \xi_1 < \xi_2 < \dots < \xi_J < 1$, $\omega_j > 0$ for all j and $\sum_j \omega_j = 1$. Then we define the corresponding composite rule

$$L_N u := h \sum_{i=0}^{N-1} \sum_{j=1}^J \omega_j u(x_i + h\xi_j) \approx \int_0^1 u(t) dt$$

and the discrete inner product

$$\langle u, v \rangle_N := L_N(u\bar{v}) \approx (u, v)_0.$$

The qualocation method to solve (5) is

$$u_h \in S_h^d, \quad \text{s.t.} \quad \langle Au_h, r_h \rangle_N = \langle f, r_h \rangle_N, \quad \forall r_h \in S_h^{d'}. \quad (6)$$

The method depends on the degrees of the spline spaces involved (d and d') and on the quadrature rule. We restrict ourselves to the case

$$d \geq n + 1.$$

With this restriction we can ensure that for all $u_h \in S_h^d$, the Fourier series of Au_h is absolutely (and hence uniformly) convergent and therefore Au_h is continuous. Part of the forthcoming analysis can be extended to cover some cases when $d = n$, namely those where Au_h can be evaluated in all the nodes $x_i + \xi_j h$ (see [8] for a discussion). For the sake of simplicity, we will omit any reference to this ‘singular’ case) (see [11] for full details).

In case S_h^0 is the test space we admit $\xi_1 = 0$ as node of the basic quadrature rule understanding that

$$r_h(x_i) := \frac{1}{2}(r_h(x_i^+) + r_h(x_i^-)), \quad r_h \in S_h^0,$$

which is the limit value of the Cauchy sums of the Fourier series of r_h at x_i .

For some requirements of our analysis we must impose the following restriction:

Hypothesis 1 *The quadrature rule satisfies that for d' even there exists $\xi_j \neq 0$ whereas for d' odd, $\xi_j \neq 1/2$ for some $1 \leq j \leq J$.*

This hypothesis will be interpreted in the following section in terms of whether the set of tests $\langle \cdot, r_h \rangle_N$, with $r_h \in S_h^{d'}$ is N -dimensional or not.

If $J = 1$ and Hypothesis 1 holds, the qualocation method is equivalent to ξ_1 -collocation

$$u_h \in S_h^d \quad \text{s.t.} \quad Au_h(x_i + \xi_1 h) = f(x_i + \xi_1 h), \quad i = 0, \dots, N - 1.$$

Conversely, we can consider ξ -collocation methods as a particular class of qualocation methods with trial spaces S_h^d where we take $d' = d$ or $d' = d + 1$ so that Hypothesis 1 is satisfied. The quadrature rule is defined by taking $J = 1$, $\xi_1 = \xi$ and $\omega = 1$.

We will extensively deal with two important sets of quadrature rules and qualocation methods which have been treated in the literature. Simpson rules are defined by taking $J = 2$, $\xi_1 = 0$ and $\xi_2 = 1/2$. Symmetric rules are those with basic quadrature points distributed symmetrically with respect to $1/2$: $\omega_r = \omega_{J-r}$, $\xi_r = 1 - \xi_{J-r}$ for all r .

4 Dual forms of the tests

For $r_h \in S_h^{d'}$ the map $\langle \cdot, r_h \rangle_N : H^1 \rightarrow \mathbb{C}$ is linear and bounded. Therefore, there exists a unique $\Pi_h r_h \in H^{-1}$ such that

$$\langle \cdot, r_h \rangle_N = (\cdot, \Pi_h r_h)_0.$$

Moreover, Π_h is a conjugate-linear map and its image $T_h := \Pi_h S_h^d$ is a finite-dimensional subset of H^s for all $s < -1/2$, since

$$T_h \subset \text{span} \langle \delta_{x_i + jh} \mid i = 0, \dots, N-1, j = 1, \dots, J \rangle$$

being δ_z the periodic Dirac delta distribution on z .

We first introduce a well-known basis of S_h^d . Let us consider the following set of representatives of Z modulo N

$$\Lambda_N := \left\{ \mu \in Z \mid -\frac{N}{2} < \mu \leq \frac{N}{2} \right\}.$$

It follows from recurrence properties of splines [4] that for $s < d' + 1/2$, there exists $C = C(s, d')$ such that

$$C \|r_h\|_s^2 \leq |\widehat{r}_h(0)|^2 + \sum_{0 \neq \mu \in \Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2 \leq \|r_h\|_s^2, \quad \forall r_h \in S_h^d. \quad (7)$$

This expression implies the existence of a set $\{\varphi_\mu^d \mid \mu \in \Lambda_N\}$ satisfying

$$\varphi_\mu^d(\nu) = \delta_{\mu\nu}, \quad \mu, \nu \in \Lambda_N,$$

where δ_μ is the Kronecker delta symbol. The set $\{\varphi_\mu^d\}$ is a basis for S_h^d , yielding

$$r_h = \sum_{\mu \in \Lambda_N} \widehat{r}_h(\mu) \varphi_\mu^d, \quad \forall r_h \in S_h^d.$$

Finally, we can use the recurrence properties of Fourier coefficients of splines to show that

$$\varphi_\mu^d = \left[1 + \Delta_d \left(N \cdot, \frac{\mu}{N} \right) \right] \phi_\mu$$

where (see [8])

$$\Delta_d(x, y) := y^{d+1} \sum_{m \neq 0} \frac{1}{(m+y)^{d+1}} \phi_m(x) : \mathbb{R} \times [-1/2, 1/2] \rightarrow \mathbb{C}.$$

Proposition 1 *The sequence of maps $\Pi_h : S_h^d \rightarrow T_h$ is uniformly bounded in H^s for all $s < -1/2$.*

Proof. Let $\mu \in \Lambda_N$ and $m \in Z$. By [8] Lemma 1 we can prove that

$$\begin{aligned} \Pi_h r_h(\mu + mN) &= (\Pi_h r_h, \phi_{\mu+mN})_0 = \langle r_h, \phi_{\mu+mN} \rangle_N \\ &= \sum_{\nu \in \Lambda_N} \widehat{r}_h(\nu) \langle \varphi_\nu^d, \phi_{\mu+mN} \rangle_N = \widehat{r}_h(\mu) c_\mu^m \end{aligned} \quad (8)$$

where

$$c_\mu^m := \langle \varphi_\mu^d, \phi_{\mu+mN} \rangle_N = \sum_{r=1}^J \omega_r \phi_{-m}(\xi_r) \left[1 + \Delta_d \left(N \cdot, \frac{\mu}{N} \right) \right]. \quad (9)$$

Since $\Delta_{\mathcal{A}}$ is bounded on $\mathbb{R} \times [-1/2, 1/2]$ we obtain that

$$|\Pi_h r_h(\mu + mN)| \leq C |\widehat{r}_h(\mu)|,$$

with C depending only on d' . Notice also that $\Pi_h r_h(0) = \widehat{r}_h(0)$. Then

$$\begin{aligned} \|\Pi_h r_h\|_s^2 &= |\widehat{r}_h(0)|^2 + \sum_{m \neq 0} |mN|^{2s} |\Pi_h r_h(mN)|^2 + \sum_{0 \neq \mu \in \Lambda_N} \sum_{m \neq 0} |\mu + mN|^{2s} |\Pi_h r_h(\mu + mN)|^2 \\ &\leq C |\widehat{r}_h(0)|^2 \left[1 + \sum_{m \neq 0} |mN|^{2s} \right] + C \sum_{0 \neq \mu \in \Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2 \left[1 + \sum_{m \neq 0} \frac{|\mu/N|^{-2s}}{|m + \mu/N|^{-2s}} \right] \\ &\leq C'_s \left[|\widehat{r}_h(0)|^2 + \sum_{0 \neq \mu \in \Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2 \right] \leq C'_s \|r_h\|_s^2, \end{aligned}$$

since $|\mu/N| \leq 1/2$ and $-2s > 1$. □

Corollary 2 *Given $s > 1/2$, there exists $C > 0$ depending on s and d' such that for all $u \in H^s$, $r_h \in S_h^d$*

$$|\langle u, r_h \rangle_N| \leq C \|u\|_s \|r_h\|_{-s}.$$

Proposition 3 *Hypothesis 1 is equivalent to Π_h having uniformly bounded inverses in H^s for all $s < -1/2$.*

Proof. We first remark that by [8], $\operatorname{Re}(1 + \Delta_{\mathcal{A}}(x, y)) > 0$ except in the points $(1/2 + k, 1/2)$, $k \in \mathbb{Z}$ for d' odd and $(k, 1/2)$ for d' even. Hence, Hypothesis 1 and the positivity of the weights ensure that

$$|c_\mu^0| \geq c > 0, \quad \forall \mu \in \Lambda_N,$$

with c_μ^0 given by (9). Then by (8) we obtain

$$C |\widehat{r}_h(\mu)| \leq |\Pi_h r_h(\mu)|, \quad \forall \mu \in \Lambda_N,$$

and consequently, by (7)

$$\|r_h\|_s \leq C_s \|\Pi_h r_h\|_s, \quad \forall r_h \in S_h^d \tag{10}$$

which proves the uniform boundedness of Π_h^{-1} .

Assume now that the hypothesis does not hold. When $\operatorname{Re}(1 + \Delta_{\mathcal{A}}(x, y)) = 0$, the imaginary part of this number is also zero, and we obtain for N even

$$c_{N/2}^m = 0, \quad \forall m \in \mathbb{Z},$$

and therefore by (8),

$$\Pi_h r_h(N/2 + mN) = 0, \quad \forall m \in \mathbb{Z}, \forall r_h \in S_h^d.$$

Taking then $r_h = \varphi_{N/2}^d$, we have $\Pi_h r_h = 0$ which makes (10) false. □

5 Asymptotics of the discrete product

In this section we work on the approximation error $\langle f, r_h \rangle_N = L_N(f\bar{r}_h) \approx (f, r_h)_0$. We begin by recalling some properties of Bernoulli functions. Let B_k be the Bernoulli polynomial of degree k ($k \geq 0$) (see [1]) and let \underline{B}_k be its 1-periodization

$$\underline{B}_k(x+m) = \underline{B}_k(x), \quad \forall x \in [0, 1), \forall m \in \mathbb{Z}.$$

It is well-known that \underline{B}_k is continuous for $k \geq 2$ since $B_k(0) = B_k(1)$. For $B_1(x) = x - 1/2$ we will define $\underline{B}_1(0) = \underline{B}_1(1) = 0$, i.e., the average of the side limits. Notice that for all $x \in \mathbb{R}$

$$\underline{B}_k(x) = (-1)^k \underline{B}_k(-x) = (-1)^k \underline{B}_k(1-x). \quad (11)$$

Moreover, \underline{B}_k has the following Fourier expansion [1]

$$\underline{B}_k := -\frac{k!}{(2\pi i)^k} \sum_{m \neq 0} \frac{1}{m^k} \phi_m.$$

We also introduce the constants

$$\gamma_k^{\mathcal{d}'} := -\frac{1}{k!} \binom{-(\mathcal{d}' + 1)}{k - (\mathcal{d}' + 1)}.$$

Lemma 4 *Let $k \in \{0, \dots, \mathcal{d}' + 1\}$. Then for all $f \in H^{k+1}$ and $r_h \in S_h^{\mathcal{d}'}$*

$$\left| h \sum_{i=0}^{N-1} f(x_i + \xi h) \overline{r_h(x_i + \xi h)} - (f, r_h)_0 \right| \leq Ch^k \|f\|_{k+1} \|r_h\|_{-1} \quad (12)$$

If $M \geq \mathcal{d}' + 1$, $f \in H^{M+2}$ and $r_h \in S_h^{\mathcal{d}'}$

$$\begin{aligned} h \sum_{i=0}^{N-1} f(x_i + \xi h) \overline{r_h(x_i + \xi h)} &= (f, r_h)_0 + \sum_{k=\mathcal{d}'+1}^M h^k (-1)^k \gamma_k^{\mathcal{d}'} \underline{B}_k(\xi) (f^{(k)}, r_h)_0 \\ &\quad + \mathcal{O}(h^{M+1}) \|f\|_{M+2} \|r_h\|_{-1}. \end{aligned} \quad (13)$$

Proof. Inequality (12) is a straightforward consequence of [13] Lemma 10. Expansion (13) follows by [13] Lemma 11. \square

Proposition 5 *There exists a sequence $(\rho_k)_k \subset \mathbb{R}$, depending on \mathcal{d}' and L , such that for all $M \geq \mathcal{d}' + 1$, $r_h \in S_h^{\mathcal{d}'}$ and $f \in H^{M+2}$*

$$(f, r_h)_0 = \langle f, r_h \rangle_N + \sum_{k=\mathcal{d}'+1}^M h^k \rho_k \langle f^{(k)}, r_h \rangle_N + \mathcal{O}(h^{M+1}) \|f\|_{M+2} \|r_h\|_{-1}. \quad (14)$$

Moreover, if the rule L is symmetric or of Simpson type then $\rho_{2k+1} = 0$ for all k .

Proof. A simple reorganization of sums in L_N gives

$$\langle u, r_h \rangle_N = \sum_{j=1}^J \omega_j \left[h \sum_{i=1}^N u(x_i + \xi_j h) \overline{r_h(x_i + \xi_j h)} \right]. \quad (15)$$

Notice that the left-hand side sum of (13) is a particular case of discrete product with a single node $\xi = \xi_1$, and that by (15) we are down to a linear combination of formulae (13) for different values of ξ to obtain a full asymptotic study of $\langle f, r_h \rangle_N$:

$$\langle f, r_h \rangle_N = (f, r_h)_0 + \sum_{k=d'+1}^M h^k (-1)^k \gamma_k^{d'} L \underline{B}_k (f^{(k)}, r_h)_0 + \mathcal{O}(h^{M+1}) \|f\|_{M+2} \|r_h\|_{-1}. \quad (16)$$

Also, by (12) we have for $k \in \{1, \dots, d' + 2\}$

$$|\langle f, r_h \rangle_N - (f, r_h)_0| \leq Ch^{k-1} \|f\|_k \|r_h\|_{-1}. \quad (17)$$

Hence, the proof of (14) is a simple reversal of expansion (16) by induction. \square

6 Consistency error expansion

We introduce a projection onto the spline space which plays a central role in our analysis. Given $u \in \mathcal{D}'$, we define

$$D_h^d u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \varphi_\mu^d \in S_h^d.$$

This operator can be understood as the result of applying a spline-trigonometric Petrov-Galerkin projection to the identity, i.e., to the trivial equation $u = u$ (see [2]). We also introduce the truncation operator for Fourier series

$$D_h^\infty u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \phi_\mu,$$

where the notation is suggested by the fact that the set of trigonometric polynomials spanned by $\{\phi_\mu \mid \mu \in \Lambda_N\}$ is the formal limit of S_h^d as d goes to infinity.

The operator D_h^d has optimal approximation properties in all available Sobolev norms [2, 13]: for $s \leq t \leq d + 1$ and $s < d + 1/2$

$$\|D_h^d u - u\|_s \leq Ch^{t-s} \|u\|_t, \quad \forall u \in H^t. \quad (18)$$

For $k \geq 1$ define

$$C_k := \mathbf{H} \underline{B}_k = -\frac{k!}{(2\pi i)^k} \sum_{m \neq 0} \frac{\text{sign}(m)}{m^k} \phi_m.$$

By (11) and since \mathbf{H} maps odd functions into even functions and viceversa, it follows that for all $k \geq 1$

$$C_k(1-x) = (-1)^{k+1} C_k(x), \quad \forall x \in \mathbb{R}. \quad (19)$$

Following [13] we denote

$$\mathcal{L}_n := \left\{ \sum_{j=1}^n a_j \mathbf{D}^j \mid a_j \in \mathcal{D} \right\},$$

the set of differential operators of order n with smooth periodic coefficients and without zero term. For $\mathbf{A} \in \mathcal{E}(n)$ with asymptotic series given by (2) we define two sequences of differential operators: for $k \geq d+1-n$

$$\mathbf{R}_k := \sum_{j=d+1}^{n+k} \gamma_k^{d+k-j} a_{j-k} \mathbf{D}^j \in \mathcal{L}_{n+k}, \quad \mathbf{T}_k := \sum_{j=d+1}^{n+k} \gamma_k^{d+k-j} b_{j-k} \mathbf{D}^j \in \mathcal{L}_{n+k}. \quad (20)$$

Following step by step the proof of [13] Proposition 3 (the norms where the expansions hold are more demanding in this new situation) we can prove that for all $M \geq d+1-n$ and $u \in H^{M+n+1}$ the following expansion holds in H^1 .

$$\mathbf{A}(D_h^d u - u) = \sum_{k=d+1-n}^{M+1} h^k \left(\underline{\mathbf{B}}_k(N \cdot) \mathbf{R}_k D_h^\infty u + C_k(N \cdot) \mathbf{T}_k D_h^\infty u \right) + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}. \quad (21)$$

Remark 6 Expansion (21) can be written without the operator D_h^∞ if $d \geq n+2$. In the case $d = n+1$, D_h^∞ can be eliminated of all the addenda except $\mathbf{T}_2 D_h^\infty u$, because of the unboundedness of C'_2 . To avoid making this distinction, we keep (21) as it is.

The quantities

$$\begin{aligned} \alpha_k &:= L\underline{\mathbf{B}}_k, & \alpha_{j,k} &:= L(\underline{\mathbf{B}}_j \underline{\mathbf{B}}_k), \\ \beta_k &:= LC_k, & \beta_{j,k} &:= L(\underline{\mathbf{B}}_j C_k), \end{aligned} \quad (22)$$

will be relevant in the sequel.

Given $\mathbf{A} \in \mathcal{E}(n)$, $d \geq n+1$ and $d' \geq 0$ we consider the following sequence of differential operators: for $k \geq d+1-n$

$$\mathbf{F}_k := \alpha_k \mathbf{R}_k + \beta_k \mathbf{T}_k + \sum_{j=d'+1}^{k-(d+1-n)} \gamma_j^{d'} (-1)^j \left[\alpha_{j,k-j} \mathbf{D}^j \mathbf{R}_{k-j} + \beta_{j,k-j} \mathbf{D}^j \mathbf{T}_{k-j} \right] \in \mathcal{L}_{n+k}. \quad (23)$$

Theorem 7 Let $\mathbf{A} \in \mathcal{E}(n)$ and $(\mathbf{F}_k)_k$ be the sequence of differential operators defined in (23). Then for all $r_h \in S_h^{d'}$ and $u \in H^{M+n+2}$

$$\langle \mathbf{A}(D_h^d u - u), r_h \rangle_N = \sum_{k=d+1-n}^M h^k (\mathbf{F}_k u, r_h)_0 + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}. \quad (24)$$

Proof. From (21) and Corollary 2 we obtain

$$\begin{aligned} \langle \mathbf{A}(D_h^d u - u), r_h \rangle_N &= \sum_{k=d+1-n}^{M+1} h^k \left[\langle \underline{\mathbf{B}}_k(N \cdot) \mathbf{R}_k D_h^\infty u, r_h \rangle_N + \langle C_k(N \cdot) \mathbf{T}_k D_h^\infty u, r_h \rangle_N \right] \\ &\quad + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}. \end{aligned}$$

On the one hand by (13)

$$\begin{aligned} \langle \underline{B}_k(N \cdot) f, r_h \rangle_N &= \sum_{r=1}^J \omega_r B_k(\xi_r) \left[h \sum_{i=0}^{N-1} f((i + \xi_r)h) r_h((i + \xi_r)h) \right] \\ &= \sum_{r=1}^J \omega_r B_k(\xi_r) \left[(f, r_h)_0 + \sum_{j=d'+1}^J h^j \gamma_j^{d'} (-1)^j \underline{B}_j(\xi_r)(f^{(j)}, r_h)_0 \right] \\ &\quad + \mathcal{O}(h^{-1}) \|f\|_{+2} \|r_h\|_{-1}. \end{aligned}$$

Let k be such that $d + 1 - n \leq k \leq M - d' - 1$. Taking $\nu = M - k$ and $f = \mathbf{R}_k D_h^\infty u$ above, we obtain the expansion

$$\begin{aligned} h^k \langle \underline{B}_k(N \cdot) \mathbf{R}_k D_h^\infty u, r_h \rangle_N &= \alpha_k h^k (\mathbf{R}_k D_h^\infty u, r_h)_0 + \sum_{j=d'+1}^{M-k} h^{j+k} \gamma_j^{d'} (-1)^j \alpha_{j,k} (\mathbf{D}^j \mathbf{R}_k D_h^\infty u, r_h)_0 \\ &\quad + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}. \end{aligned} \quad (25)$$

For $M - d' \leq k \leq M + 1$ we use (12) to obtain

$$h^k |\langle \underline{B}_k(N \cdot) \mathbf{R}_k D_h^\infty u, r_h \rangle_N - \alpha_k (\mathbf{R}_k D_h^\infty u, r_h)_0| \leq C h^{M+1} \|u\|_{M+n+2} \|r_h\|_{-1}. \quad (26)$$

Similar arguments can be applied to prove that for $d + 1 - n \leq k \leq M - d' - 1$

$$\begin{aligned} h^k \langle C_k(N \cdot) \mathbf{T}_k D_h^\infty u, r_h \rangle_N &= h^k \beta_k (\mathbf{T}_k D_h^\infty u, r_h)_0 + \sum_{j=d'+1}^{M-k} h^{j+k} \gamma_j^{d'} (-1)^j \beta_{j,k} (\mathbf{D}^j \mathbf{T}_k D_h^\infty u, r_h)_0 \\ &\quad + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}, \end{aligned} \quad (27)$$

whereas for $M - d' \leq k \leq M + 1$ a bound similar to (26) holds. Collecting these expansions, we obtain

$$\langle \mathbf{A}(D_h^d u - u), r_h \rangle_N = \sum_{k=d+1-n}^M h^k (\mathbf{F}_k D_h^\infty u, r_h)_0 + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}.$$

The bound

$$|(\mathbf{F}_k(u - D_h^\infty u), r_h)_0| \leq C \|u - D_h^\infty u\|_{n+k+1} \|r_h\|_{-1} \leq C' h^{M+1-k} \|u\|_{M+n+2},$$

derived from the order of \mathbf{F}_k and (18) finishes the proof. \square

The next result is now a simple consequence of Theorem 7 and Proposition 5.

Corollary 8 *Let $\mathbf{A} \in \mathcal{E}(n)$ and $d \geq n + 1$. Then there exists a sequence $(\mathbf{G}_k)_k$ with $\mathbf{G}_k \in \mathcal{L}_{n+k}$ satisfying the following property: for all $r_h \in S_h^{d'}$ and $u \in H^{M+n+2}$*

$$\langle \mathbf{A}(D_h^d u - u), r_h \rangle_N = \sum_{k=d+1-n}^M h^k \langle \mathbf{G}_k u, r_h \rangle_N + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}. \quad (28)$$

Remark 9 For the particular case of ξ -collocation methods, i.e., for single point rules, Corollary 8 is a direct consequence of (21) taking

$$\mathbf{G}_k = \alpha_k \mathbf{R}_k + \beta_k \mathbf{T}_k = \underline{B}_k(\xi) \mathbf{R}_k + C_k(\xi) \mathbf{T}_k. \quad (29)$$

In the sequel, the cancellation of some of the first \mathbf{G}_k will be relevant, since it provokes an improvement in the order of convergence of the method.

Definition 1 Let $b \geq 0$ be the greatest integer such that

$$\mathbf{G}_k = 0, \quad k = d + 1 - n, \dots, d + b - n \quad (30)$$

($b = 0$ means no cancellation of the first operator). We call this number, depending on d , d' , L and \mathbf{A} , the additional order of convergence of the method.

Notice that if $\mathbf{F}_j = 0$ for $d + 1 - n \leq j \leq k$ then $\mathbf{G}_j = 0$ for j in the same range. Since the expression of the operators \mathbf{F}_j is simpler than that of \mathbf{G}_j , we will study the additional order by means of the cancellation of \mathbf{F}_k instead of using (30). A priori, the additional order is achieved for a single equation $\mathbf{A}u = f$, although in most cases it will be proven for classes of operators having similar expansions.

7 Stability

We begin this section by writing the qualocation method in a standard projection form. To do that we introduce the artificial discrete projection P_h which assigns to $u \in H^s$ $s > 1/2$ the unique solution of

$$P_h u \in S_h^{d'+2}, \quad \text{s.t.} \quad \langle P_h u, r_h \rangle_N = \langle u, r_h \rangle_N, \quad \forall r_h \in S_h^d.$$

Therefore, P_h is the qualocation operator associated to the identity. We use $d' + 2$ to include the case $d' = 0$; otherwise we would be allowed to take $P_h u \in S_h^{d'}$ without further complication. Then

$$\|P_h u\|_1 \leq C \|u\|_1, \quad \|P_h u - u\|_1 \leq C h^{d'+1} \|u\|_{d'+2}.$$

Obviously u_h is the solution to

$$u_h \in S_h^d, \quad \text{s.t.} \quad \langle \mathbf{A}u_h, r_h \rangle_N = \langle \mathbf{A}u, r_h \rangle_N, \quad \forall r_h \in S_h^d,$$

if and only if $P_h \mathbf{A}u_h = P_h \mathbf{A}u$. Let then

$$A_h := P_h \mathbf{A}|_{S_h^d} : S_h^d \rightarrow S_h^{d'+2}$$

which is a sequence of uniformly bounded operators from H^{n+1} to H^1 . Moreover, if the qualocation equations have a unique solution then

$$Q_h := A_h^{-1} P_h \mathbf{A}$$

is the operator mapping the exact to the numerical solution.

Proposition 10 *The following statements are equivalent:*

(a) *The qualocation method is H^{n+1} -stable, i.e.,*

$$\|Q_h u\|_{n+1} \leq C \|u\|_{n+1}, \quad \forall u \in H^{n+1}.$$

(b) *The uniform Babuška-Brezzi condition holds*

$$\inf_{0 \neq g_h \in S_h^d} \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle \mathbf{A}g_h, r_h \rangle_N|}{\|g_h\|_{n+1} \|r_h\|_{-1}} \geq \beta > 0. \quad (31)$$

(c) *The family A_h has uniformly bounded inverses, i.e.*

$$\|g_h\|_{n+1} \leq C \|A_h g_h\|_1, \quad \forall g_h \in S_h^d.$$

Proof. It is obvious that (c) implies (a). Let then $P_h^* : H^{-1} \rightarrow H^{-1}$ be the 0-adjoint operator to $P_h : H^1 \rightarrow H^1$,

$$(P_h^* v, u)_0 = (v, P_h u)_0, \quad \forall u \in H^1, v \in H^{-1}.$$

If $t_h \in T_h$ we have that

$$(u, P_h^* t_h)_0 = (P_h u, t_h)_0 = (u, t_h)_0$$

(recall that $T_h = \Pi_h S_h^d$ and the definition of Π_h in section 4). Therefore $P_h^* t_h = t_h$ for all $t_h \in T_h$. Since by transposition the image of P_h^* is at most N -dimensional, it follows that P_h^* is a projection onto T_h .

Assume now that (a) holds. Given $0 \neq u_h \in S_h^d$ we define $w_h = P_h \mathbf{A} u_h$. Since

$$P_h \mathbf{A} (\mathbf{A}^{-1} w_h) = w_h = P_h \mathbf{A} u_h, \quad u_h \in S_h^d,$$

then $Q_h (\mathbf{A}^{-1} w_h) = u_h$. Therefore, (a) implies

$$\|u_h\|_{n+1} \leq C \|\mathbf{A}^{-1} w_h\|_{n+1} \leq C' \|w_h\|_1. \quad (32)$$

Taking into account the identities

$$(w_h, v)_0 = (P_h w_h, v)_0 = (w_h, P_h^* v)_0, \quad (w_h, \Pi_h r_h)_0 = \langle P_h \mathbf{A} u_h, r_h \rangle_N = \langle \mathbf{A} u_h, r_h \rangle_N$$

for all $v \in H^{-1}$ and $r_h \in S_h^{d'}$ we have

$$\begin{aligned} \|w_h\|_1 &= \sup_{0 \neq v \in H^{-1}} \frac{|(w_h, v)_0|}{\|v\|_{-1}} \leq C \sup_{0 \neq P_h^* v \in H^{-1}} \frac{|(w_h, P_h^* v)_0|}{\|P_h^* v\|_{-1}} \\ &= C \sup_{0 \neq r_h \in S_h^{d'}} \frac{|(w_h, \Pi_h r_h)_0|}{\|\Pi_h r_h\|_{-1}} \leq C' \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle \mathbf{A} u_h, r_h \rangle_N|}{\|r_h\|_{-1}} \end{aligned} \quad (33)$$

where we have used the uniform boundedness of P_h^* and Π_h^{-1} . Obviously, (32) and (33) imply the Babuška-Brezzi condition.

If (b) holds, then

$$\beta \|u_h\|_{n+1} \leq \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle \mathbf{A}u_h, r_h \rangle_{\mathcal{N}}|}{\|r_h\|_{-1}} = \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle A_h u_h, r_h \rangle_{\mathcal{N}}|}{\|r_h\|_{-1}} \leq C \|A_h u_h\|_1,$$

by Corollary 2, which proves (c). \square

As a consequence of the preceding result we can easily prove the following lemma, showing preservation of stability under compact perturbations and multiplication operators.

Lemma 11 *Let $\mathbf{A} \in \mathcal{E}(n)$ be invertible and such that the qualocation method is stable. Let $\mathbf{K} : H^s \rightarrow H^{s-n}$ be compact for all s , such that $\mathbf{A} + \mathbf{K}$ is injective and $a \in \mathcal{D}$ satisfying $a(t) \neq 0$ for all t . Then the qualocation method for $\mathbf{A} + \mathbf{K}$ and $a\mathbf{A}$ is stable.*

Proof. The first result is a simple consequence of characterization (c) of stability with standard techniques (see [15] Chapter 13 and [16] Chapter 1).

To see the second one, notice first that if $\mathbf{A} \in \mathcal{E}(n)$ then by (3) and (4), $\mathbf{B} := a\mathbf{A} - \mathbf{A}a \in \mathcal{E}(n-1)$ is compact as an operator of order n . Let then

$$A_h := P_h \mathbf{A}|_{S_h^d}, \quad \tilde{A}_h := P_h \mathbf{A}a|_{S_h^d}.$$

By a commutator property of splines (see [24] Theorem 2.1) we have for all $u_h \in S_h^d$

$$\|D_h^d(au_h) - au_h\|_{n+1} \leq Ch \|u_h\|_{n+1}. \quad (34)$$

Then

$$\|A_h D_h^d(au_h) - \tilde{A}_h u_h\|_1 \leq C'h \|u_h\|_{n+1}. \quad (35)$$

Thus for h small enough

$$\begin{aligned} \|u_h\|_{n+1} &\leq C_1 \|au_h\|_{n+1} \leq C_1 \|D_h^d(au_h)\|_{n+1} + C_2 h \|u_h\|_{n+1} \\ &\leq C_3 \|A_h D_h^d(au_h)\|_1 + C_2 h \|u_h\|_{n+1} \leq C_3 \|\tilde{A}_h u_h\|_1 + C_4 h \|u_h\|_{n+1}, \end{aligned}$$

where we have successively applied: the invertibility of the multiplication operator, the commutator property (34), the stability of the qualocation methods for \mathbf{A} and (35). By Proposition 10, this proves that the method is stable for $\mathbf{A}a$. By the first result of this Lemma and the compactness of \mathbf{B} , the method is stable for $a\mathbf{A}$. \square

8 Full expansion of the error and some consequences

We will prove in this part an error expansion in powers of h of the difference between the numerical solution and the Fourier spline projection of the exact solution. To prove it, we will use the infimum-supremum condition of Proposition 10 to transform the consistency expansion stated in section 6 into this error expansion. As by-products we obtain an estimation of the order of the method, pointwise superconvergence in some particular points of the grid and a true asymptotic expansion, i.e., with the coefficients independent of h , of the error under the action of smooth operators.

Proposition 12 *Given $\mathbf{A} \in \mathcal{E}(n)$ and a stable quadrature method with additional order b , there exists a sequence of pseudodifferential operators $(\mathbf{Q}_k)_k$ with $\mathbf{Q}_k \in \mathcal{E}(k)$ such that the following expansion holds in H^{n+1}*

$$D_h^d u - Q_h u = \sum_{k=d-n+b+1}^M h^k D_h^d \mathbf{Q}_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}. \quad (36)$$

Proof. From Theorem 7 and Corollary 2, we deduce

$$\left\langle \mathbf{A}(D_h^d u - Q_h u - \sum_{k=d-n+b+1}^M h^k Q_h \mathbf{A}^{-1} \mathbf{G}_k u), r_h \right\rangle_N = \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}.$$

The infimum-supremum condition (31) yields

$$D_h^d u - Q_h u = \sum_{k=d-n+b+1}^M h^k Q_h \mathbf{A}^{-1} \mathbf{G}_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}$$

in H^{n+1} . Notice that we can apply the same argument to each term $Q_h \mathbf{A}^{-1} \mathbf{G}_k u$. Hence the result follows readily by induction. \square

From Proposition 12 and (18) we obtain

$$\|Q_h u - u\|_{n-b} \leq \|Q_h u - D_h^d u\|_{n+1} + \|D_h^d u - u\|_{n-b} \leq Ch^{d-n+b+1} \|u\|_{d+b+2}. \quad (37)$$

Therefore, we conclude that the order of the method is $d - n + b + 1$ in the norm $\|\cdot\|_{n-b}$, whenever the exact solution is smooth enough. Hence the order of the method equals the first power of h appearing in (36)

It is well known that for $n \geq 1$, the Bernoulli polynomial B_{2n} has only two roots ζ and $1 - \zeta$ both in $(0, 1)$. For n odd, the roots of B_{2n+1} are $\{0, 1/2, 1\}$. Taking into account the exception of how we evaluate \underline{B}_1 at its breakpoints, we can say that \underline{B}_n has two different roots in each period. We denote them ζ_1 and ζ_2 .

Proposition 13 *Assume that $b \geq \max\{n, 1\}$. Then*

$$\max_i |Q_h u(x_i + \zeta_j h) - u(x_i + \zeta_j h)| \leq Ch^{d+2} \|u\|_{d+b+2}, \quad j = 1, 2.$$

Proof. In [12] Corollary 3.2 it is proven that

$$\max_i |D_h^d u(x_i + \zeta_j h) - u(x_i + \zeta_j h)| \leq Ch^{d+2} \|u\|_{d+3}.$$

If $n \leq -1$ the inverse inequalities for splines and (36) with $M = d - n + b$ yield

$$\|D_h^d u - Q_h u\|_\infty \leq Ch^{n+1/2} \|D_h^d u - Q_h u\|_{n+1} \leq C' h^{d+b+3/2} \|u\|_{d+b+2},$$

which proves the result for this case. When $n \geq 0$, we simply apply (36) and Sobolev's embedding theorem. \square

We finish this section by stating the existence of an asymptotic expansion of the error between the exact and numerical solution under the action of a smoothing operator. Such operators appear, for instance, in the boundary integral methods when we want to recover the solution of the differential problem by means of an integral representation formula or potential.

Proposition 14 *Let X be a normed space and $T : \mathcal{D}' \rightarrow X$ be bounded in the norm of H^s for all s . Then*

$$T(u - Q_h u) = \sum_{k=d-n+b+1}^M h^k T Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}.$$

9 Particular equations and quadrature rules

In this section we will study the error expansion (36) of particular methods applied to some equations arising from practical situations. Furthermore, since we will give an exact expression of these terms, we will be able to deduce sufficient conditions to obtain quadrature rules which define methods of higher order by demanding the cancellation of the first terms of these expansions.

We remark that these conditions and the rules derived therefrom are already known and were stated in [8, 23, 24], although we arrive at the same results from a different way. Our approach enables to prove that some methods keep their orders of convergence when they are applied to more general equations. For instance, we will extend the good properties of the Simpson rules to logarithmic, second kind and hypersingular equations with variable-coefficients. Moreover, we can prove that the first power of h appearing in the error expansion is multiplied by, in general, a non zero function. This implies that the order proved in (37) is optimal. On the other hand our smoothness requirements over the exact solution are a little stronger than those in [8, 23, 24].

We will pay attention to four cases:

- (a) collocation in midpoints or nodes,
- (b) ξ -collocation with $\xi \neq 0, 1/2$,
- (c) qualocation methods with Simpson rules,
- (d) qualocation methods with symmetric rules.

Since we are interested in studying when the operators F_k are null, we are concerned about coefficients $\alpha_j, \beta_j, \alpha_{j,k}$ and $\beta_{j,k}$ appearing in the definition of these operators. By Remark 9 in cases (a) and (b) only α_j and β_j have to be considered.

Lemma 15 *In cases (a), (c) and (d)*

$$\alpha_{2k+1} = \beta_{2k} = 0, \quad \forall k \in \mathbb{Z}. \quad (38)$$

For Simpson rules (case (c)) it holds that

$$\begin{cases} \alpha_{j,k} = 0, & \text{if } j \text{ or } k \text{ is odd,} \\ \beta_{j,k} = 0, & \text{if } j \text{ is odd or } k \text{ is even,} \end{cases} \quad (39)$$

whereas for symmetric rules (d) we have

$$\begin{cases} \alpha_{j,k} = 0, & \text{if } j+k \text{ is odd,} \\ \beta_{j,k} = 0, & \text{if } j+k \text{ is even.} \end{cases} \quad (40)$$

Proof. It is a straightforward consequence of (11) and (19).
We will need to deal with the roots of C_{2k} . Similar to B

□

- (b) By Remark 9 the first term in expansion (28) is related to the operator $\mathbf{G}_{d+2} = C_{d+2}(\xi)\mathbf{T}_{d+2}$. If d odd and ξ is one of the two roots of C_{d+2} , the ξ -collocation method is of order $d + 3$, but with error expansion (28) containing even and odd powers of h . For other choices of ξ and d the method is of order $d + 2$.
- (c) Applying (38) and (39) we obtain that $\mathbf{F}_{2k} = 0$ for all k and expansion (24) has only odd powers of h . By Proposition 5 this property is kept in (28), i.e, $\mathbf{G}_{2k} = 0$ for all k . Thus, Simpson rules define methods of order at least $p(d) + 1$. Moreover, since $\mathbf{G}_{p(d)+1} = \beta_{p(d)+1}\mathbf{T}_{p(d)+1}$, then by (41), $\beta_{p(d)+1} = 0$ if

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)} - 1}{2^{p(d)+1} - 1}.$$

The Simpson rule with these weights defines a method of order $p(d) + 3$. The error expansion in this case is

$$D_h^d u - Q_h u = \sum_{k=p(d)/2+1}^{p(d)+2} h^{2k+1} D_h^d Q_{2k} u + \sum_{k=2p(d)+6}^M h^k D_h^d Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+1}. \quad (44)$$

- (d) As in (c), we obtain from (38), (40) and Proposition 5 that $\mathbf{F}_{2k} = \mathbf{G}_{2k} = 0$ for all k . Therefore expansions (24) and (28) have only odd powers of h . If b is such that $d + b + 1$ is even and

$$\mathbf{F}_{2k+1} = 0 \quad k = p(d)/2, \dots, (d + b - 1)/2,$$

then the method has additional order b . These cancellation of operators are for instance obtained if

$$\beta_{2k+1} = \beta_{j, 2k+1-j} = 0, \quad k = \frac{p(d)}{2}, \dots, \frac{d + b - 1}{2}, \quad d' + 1 \leq j \leq 2k - d - 1.$$

There is a simple choice [8] with a two-point symmetric rule if $d' \geq 1$, since in this case

$$\mathbf{G}_{p(d)+1} = \mathbf{F}_{p(d)+1} = \beta_{p(d)+1} \mathbf{T}_{p(d)+1}.$$

Taking ξ_1, ξ_2 the roots of $C_{p(d)}$ and $\omega_1 = \omega_2 = 1/2$ we have that $\beta_{p(d)+1} = 0$, from where the order of the method becomes $p(d) + 3$. In the general case discussed above, this corresponds to $b = 2$ or 3 depending on whether d is odd or even respectively. When $d' = 0$ (and hence we take d even) the operator $\mathbf{G}_{p(d)+1} = \mathbf{F}_{p(d)+1}$ involves the coefficients β_{d+3} and $\beta_{1, d+2}$. Therefore cancellation of these is not so simple.

Equations of the second kind. Let $a \in \mathcal{D}$ be such that $a(t) \neq 0$ for all t , and $K \in \mathcal{C}^\infty(\mathbb{R}^2)$ be 1-periodic in both variables. We consider equations of the form

$$\mathbf{A}u := au + \int_0^1 K(\cdot, t)u(t) dt = f. \quad (45)$$

The operator has the trivial expansion $\mathbf{A} \stackrel{\text{exp}}{=} aD^0$, and therefore $\mathbf{T}_k = 0$, $\mathbf{R}_k = \gamma_k^d a D^k$, for $k \geq d + 1$. Cases (c) and (d) are stable if d and d' are of the same parity, whereas the unique unstable ξ -collocation methods are the collocation in midpoints for d odd and knot collocation for d is even.

We now study the order of convergence and the error expansion.

- (a) Using Remark 9 and (38) we conclude that $\mathbf{G}_{2k+1} = 0$ for all k . Moreover, by Proposition 12 we can prove that $\mathbf{Q}_{2k+1} = 0$ and expansion (36) only has even powers of h . Thus stable collocation methods are of order $p(d)$
- (b) By (29), the first term in expansion (28) is related to $\mathbf{G}_{d+1} = B_{d+1}(\xi)\mathbf{R}_{d+1}u$. For d odd and ξ one of the two roots of B_{d+1} , the corresponding ξ -collocation method is of order $d + 2$. Other choices of ξ and d give methods of order $d + 1$.
- (c) Applying (38) and (40) we obtain that $\mathbf{F}_{2k+1} = 0$ for all k and expansion (24) only has odd powers of h . By Proposition 5 it holds that in expansion (28) also $\mathbf{G}_{2k+1} = 0$. Now it is straightforward to verify that expansion (36) has only even powers of h . On the other hand, the first power appearing in (28) is $p(d)$. From (41) we conclude that taking

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)-1} - 1}{2^{p(d)} - 1},$$

$\alpha_{p(d)} = 0$ and therefore $\mathbf{G}_{p(d)} = \alpha_{p(d)}\mathbf{R}_{p(d)} = 0$. Hence this method has order $p(d) + 2$.

- (d) As in (c), we obtain that in expansions (24), (28) and (36) there are no odd powers of h . Now the method is of even order $d + b + 1$ if

$$\alpha_{2k} = \alpha_{j,2k-j} = 0, \quad d' + 1 \leq j \leq 2k - d - 1, \quad k = p(d)/2, \dots, (d + b - 1)/2,$$

because of these cancellations imply that $\mathbf{F}_{2k} = 0$ for $k = p(d)/2, \dots, (d + b - 1)/2$. If $d' \geq 1$ the symmetric rule of two points with ξ_1, ξ_2 the roots of $B_{p(d)}$ and the weights $\omega_1 = \omega_2 = 1/2$ gives a method of order $p(d) + 2$ with only even powers of h in expansion (36).

Equations of the second kind with a logarithmic part. Let

$$\mathbf{A}u := au + \int_0^1 A(\cdot, t) \log(\sin^2 \pi(\cdot - t))u(t) dt + \int_0^1 K(\cdot, t)u(t) dt = f.$$

Then, \mathbf{A} has the asymptotic expansion

$$\mathbf{A} \stackrel{\text{exp}}{=} aD^0 + \sum_{j \leq -1} b_j \mathbf{H}D^j.$$

Thus, these equations are compact perturbations of equations of second kind and therefore, the requirements for ensuring stability are the same than in this case.

Collocation and ξ -collocation methods keep their order of convergence, but with even and odd powers of h appearing in their error expansions.

On the other hand, qualocation methods corresponding to Simpson rules and the symmetric rule with two points have now order $p(d) + 1$. The conditions to obtain qualocation methods of higher order can be deduced following similar arguments to those applied above. These conditions involve now the coefficients $\alpha_{2k}, \beta_{2k+1}, \alpha_{j,2k-j}$ and $\beta_{j,2k+1-j}$ with k in a suitable range, which depends on the required order.

Singular integral equations. We consider equations of the form

$$Su = au + bHu + Ku = f,$$

being K a ψ do of order $-\infty$. The stability of collocation and ξ -collocation methods depends now on the balance between a and b and on the choice of the point of collocation (see [19] Theorem 3). On the other hand, the stability of qualocation methods is not so clear, and cannot be deduced by the same ideas applied so far. We refer to [24] for a review of stability conditions for qualocation methods, and to [18] for very recent contributions to this problem.

Some hypersingular equations. Let

$$\begin{aligned} Wu := & \text{f.p.} \int_0^1 A_1(\cdot, t) \frac{1}{\sin^2(\pi(\cdot - t))} dt + \text{p.v.} \int_0^1 A_2(\cdot, t) \cot(\pi(\cdot - t))u(t) dt + \\ & + \int_0^1 A_3(\cdot, t) \log(\sin^2 \pi(\cdot - t))u(t) dt + \int_0^1 K(\cdot, t)u(t) dt = f, \end{aligned}$$

where f.p. denotes the Hadamard finite part, and p.v, the Cauchy principal value. Here we must impose that $A_1(t, t) \neq 0$ for all t . By [13] Appendix,

$$W \stackrel{\text{exp}}{=} \sum_{j \leq 1} b_j HD^j \in \mathcal{E}(1),$$

with $b_1(t) = (\pi t)^{-1} A(t, t)$. Therefore, for d and d' of the same parity the unique unstable qualocation methods are those equivalent to collocation in midpoints for d odd and in the nodes for d even.

The stable collocation method gives the expansion

$$D_h^d u - Q_h u = \sum_{k=p(d)/2-1}^{p(d)-2} h^{2k+1} D_h^d Q_{2k+1} u + \sum_{k=2p(d)-2}^M h^k D_h^d Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+3}$$

Again, for d odd there exist two points ξ_1, ξ_2 (the roots of C_d) such that the corresponding ξ -collocation is of order $d + 1$.

The Simpson rule with

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)-2} - 1}{2^{p(d)-1} - 1}$$

and the symmetric rule of two points ξ_1, ξ_2 with $d' \geq 1$ define methods with order $p(d) + 1$. The error expansion is now

$$D_h^d u - Q_h u = \sum_{k=p(d)/2}^{p(d)} h^{2k+1} D_h^d Q_{2k+1} u + \sum_{k=2p(d)+2}^M h^k D_h^d Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+3}.$$

10 An example with numerical integration

In this section we illustrate with an example the convergence properties of the qualocation method of Simpson type with piecewise functions as trial and test space for logarithmic equations. We show how this can be implemented with numerical integration applied in a way that preserves the order and the error expansion of the method. We remark that similar strategies can be developed to cover all qualocation methods for this kind of equations and can be extended to equations of second kind with logarithmic part.

Let us consider the equation

$$\mathbf{V}u := \int_0^1 V(\cdot, t)u(t) dt = \int_0^1 A(\cdot, t) \log(\sin^2 \pi(\cdot - t))u(t) dt + \int_0^1 K(\cdot, t)u(t) dt = f,$$

where $A, K \in \mathcal{C}^\infty(\mathbb{R}^2)$ are 1-periodic in both variables. We consider the basis $\{\chi_i\}_{i=0}^{N-1}$ of S_h^0 , being χ_i the 1-periodization of the characteristic function of the interval (x_i, x_{i+1}) . Then the $S_h^0 \times S_h^0$ qualocation method is equivalent to solving

$$\sum_{j=0}^{N-1} u_j \langle \mathbf{V}\chi_j, \chi_i \rangle_N = \langle f, \chi_i \rangle_N, \quad i = 0, \dots, N-1.$$

Notice that if $b_{i,j} = \langle \mathbf{V}\chi_j, \chi_i \rangle_N$ and $a_{i,j}(\xi) = \mathbf{V}\chi_j(x_i + \xi h)$ then

$$b_{i,j} = h \left[\frac{1}{2}a_{i,j}(0) + a_{i,j}\left(\frac{1}{2}\right) + \frac{1}{2}a_{i+1,j}(0) \right]. \quad (46)$$

To approximate $a_{i,j}(\xi)$ we follow a variant of the strategies of [7], themselves based upon [14]. We first choose $C \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that

$$C(s+1, t+1) = C(s, t), \quad \frac{\partial^k C}{\partial t^k}(s, s) = \frac{\partial^k A}{\partial t^k}(s, s), \quad k = 0, 1, 2, \quad (47)$$

and remark that

$$F(s, t) := V(s, t) - C(s, t) \log(s-t)^2$$

admits a \mathcal{C}^2 extension to the diagonal $s = t$. For (i, j) such that $|i - j| \leq N/2$ we approximate

$$\alpha_{i,j}^{(1)}(\xi) := \int_{z_j - h/2}^{z_j + h/2} F(x_i + \xi h, t) dt \approx h \text{LF}(x_i + \xi h, z_j + h \cdot) =: \beta_{i,j}^{(1)}(\xi)$$

where $z_i = x_i + h/2$ and the quadrature rule

$$Lu := \frac{1}{5760} [-17u(-2) + 308u(-1) + 5178u(0) + 308u(1) - 17u(2)] \approx \int_{-1/2}^{1/2} u(t) dt,$$

is exact for polynomials of degree not greater than 5. For the same pairs (i, j) we compute $\Pi_{i,j} \in \mathbb{P}_4$ such that

$$\Pi_{i,j}(z_j + kh) = C(x_i + \xi h, z_j + kh), \quad k = -2, -1, 0, 1, 2$$

and approximate

$$\begin{aligned}\alpha_{ij}^{(2)}(\xi) &:= \int_{z_j-h/2}^{z_j+h/2} C(x_i + \xi h, t) \log(x_i + \xi h - t)^2 dt \\ &\approx \int_{z_j-h/2}^{z_j+h/2} \Pi_{ij}(t) \log(x_i + \xi h - t)^2 dt =: \beta_{ij}^{(2)}(\xi).\end{aligned}$$

The last quantities can be computed exactly (cf. [7, 11]). Finally we add $\beta_{ij}(\xi) := \beta_{ij}^{(1)}(\xi) + \beta_{ij}^{(2)}(\xi)$, and reorganise

$$a_{ij}^*(\xi) := \begin{cases} \beta_{ij}(\xi), & \text{if } |i-j| < N, \\ \beta_{ij+N}(\xi), & \text{if } i-j > N/2, \\ \beta_{ij-N}^{(k)}(\xi), & \text{if } j-i > N/2, \\ \frac{1}{2}(\beta_{ij-N}^{(k)}(\xi) + \beta_{ij+N}^{(k)}(\xi)), & \text{if } |i-j| = N/2. \end{cases}$$

Hence (46) can be used to give an approximation $b_{ij}^* \approx b_{ij}$ to the matrix of our system. Notice that we have only used values of F and C at points $(x_i + \xi h, z_j)$, $|i-j| \leq N/2 + 2$.

Let now u_h^* be the solution of the discrete qualocation method. The analysis given in [7] can be adapted to prove H^0 -stability of the numerical method and that the following expansion

$$D_h^0 u - u_h^* - \sum_{k=5}^M h^k D_h^0 u_k^* = \mathcal{O}(h^{M+1})$$

holds in L^∞ . Here, the functions u_k^* depend on the operator \mathbf{V} and on the exact solution u . Notice that in particular the order of the method is preserved.

Example. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ . We take $\mathbf{x} : [0, 1] \rightarrow \Gamma$ a regular \mathcal{C}^∞ parameterization of Γ . Then it is well-known (cf. [9]) that the solution of

$$\begin{cases} -\Delta \omega + \lambda^2 \omega = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma, \\ \omega|_\Gamma = f, \\ -\frac{\omega}{r}(\mathbf{y}) + \lambda \omega(\mathbf{y}) = o(|\mathbf{y}|^{-1/2}), \text{ as } |\mathbf{y}| \rightarrow \infty, \end{cases} \quad (48)$$

can be written in the form

$$\omega(\mathbf{y}) = \int_0^1 K_0(\lambda|\mathbf{y} - \mathbf{x}(t)|)u(t) dt, \quad \forall \mathbf{y} \in \mathbb{R}^2 \setminus \Gamma,$$

if λ^2 is not a Dirichlet eigenvalue of the Laplace operator in Ω . Here K_0 is the modified Bessel function of the second kind and order 0 and u must be the solution of the periodic integral equation

$$\int_0^1 K_0(\lambda|\mathbf{x}(\cdot) - \mathbf{x}(t)|)u(t) dt = f(\mathbf{x}(\cdot)). \quad (49)$$

The kernel satisfies the decomposition (see [1])

$$K_0(\lambda|\mathbf{x}(s) - \mathbf{x}(t)|) = -\frac{1}{2}I_0(\lambda|\mathbf{x}(s) - \mathbf{x}(t)|) \log(\sin^2 \pi(s-t)) + B(s, t).$$

with $I_0 \in \mathcal{C}^\infty(\mathbb{R})$ is the modified Bessel function of the first kind and order 0 and B smooth. Therefore, equation (49) is of logarithmic type. We take

$$C(s, t) := -\frac{1}{2} - \frac{\lambda^2}{8} |\mathbf{x}'(s)|^2 (s - t)^2,$$

that satisfies (47). Notice that with this decomposition, the polynomials Π_{ij} appearing in the definition of $\beta_{ij}^{(2)}(\xi)$ are of degree 2. If $u_h^* = \sum_{i=0}^{N-1} u_i^* \chi_i$ is the numerical solution, we define

$$\omega_h(\mathbf{y}) := h \sum_{i=0}^{N-1} u_i^* \mathbb{L}[K_0(\lambda|\mathbf{y} - \mathbf{x}(x_i + h \cdot)|)] \approx \omega(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2 \setminus \Gamma,$$

the approximate solution of problem (48). Using now Proposition 14 and [7] Theorem 8.1 we can prove that there exists a sequence of functions $(v_k)_k$ such that

$$\omega(\mathbf{y}) - \omega_h(\mathbf{y}) = \sum_{k=5}^M h^k v_k(\mathbf{y}) + \mathcal{O}(h^{M+1})$$

uniformly over compact sets of $\mathbb{R}^2 \setminus \Gamma$.

If the origin is contained in Ω , then for $f = K_0(\lambda|\mathbf{y}|)$ the unique solution of (48) in $\Omega' := \mathbb{R}^2 \setminus \Omega$ is $\omega(\mathbf{y}) := K_0(\lambda|\mathbf{y}|)$. In our experiment we have taken the ellipse Γ

$$\mathbf{x}(t) := (0.25 + 2 \cos(2\pi t), 0.4 + \sin(2\pi t)),$$

$\lambda = 1/2$ and $N = 64, 96, 144, 216, 324, 486$.

We approximate the solution of the differential problem in $\mathbf{y}_0 = (2, 3)$, where $\omega(\mathbf{y}_0) \approx 0.145425$. Notice that the ratio of discretization between two consecutive grids is $3/2$. Then, using Richardson extrapolation (cf. [25]), we can define a new set of numerical solutions

$$\begin{cases} \omega_h^0 := \omega_h(\mathbf{y}_0) \\ \omega_h^j := \omega_h^0 \left(\frac{3}{2}\right)^j \end{cases}$$

Table 1: Errors after extrapolation

N	ϵ_h^0	ϵ_h^1	ϵ_h^2	ϵ_h^3	ϵ_h^4	ϵ_h^5
64	5.30E-09					
96	7.34E-10	4.19E-11				
144	1.00E-10	3.98E-12	3.28E-13			
216	1.35E-11	3.66E-13	1.81E-14	1.23E-15		
324	1.81E-12	3.31E-14	1.01E-15	4.59E-17	2.17E-18	
486	2.41E-13	2.95E-15	5.77E-17	1.73E-18	6.05E-20	4.20E-21

Table 2: E.c.r. for the errors in Table 1

	5	6	7	8	9
	4.874				
	4.913	5.807			
	4.941	5.886	7.153		
	4.961	5.930	7.103	8.110	
	4.974	5.956	7.069	8.081	8.827

The qualocation method can then be written as

$$u_h \in S_h^d, \quad \text{s.t.} \quad \langle Au_h, r_h \rangle_N = \langle f, r_h \rangle_N, \quad \forall r_h \in S_h^\infty.$$

We remark that hypothesis 1 makes no sense now since $d' = \infty$ is neither even nor odd. Moreover, Proposition 1 (and hence Corollary 2) holds and Proposition 3 is always satisfied. To see that this is so we just only have to notice that the proof is valid by taking $\Delta_\infty = 0$.

Again it is possible to prove that (17) holds for all k , which makes the approximate inner product asymptotically optimal. The proof follows readily by making the same trick (take $\Delta_\infty = 0$) in the proof of [13] Lemma 10. Consequently, the proof of Theorem 7 is greatly simplified, and the expansion holds with $F_k = \alpha_k R_k + \beta_k T_k$ for all k . Moreover, by the good approximation properties of the discrete inner product, Corollary 8 holds with $G_k = F_k$. It is also straightforward that Proposition 12 and 13 are also valid.

With respect to stability, the proof of Lemma 16 can be adapted to obtain that the qualocation method is stable for that kind of operators if and only if they are not equivalent to ill posed collocation: midpoint collocation for d odd and A strongly elliptic or d even and A oddly elliptic and knot collocation for d even and A strongly elliptic or d odd and A oddly elliptic. For the particular equations examined in section 9, and the methods studied therein, the orders of convergence and the error expansions are also valid.

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