High order collocation and quadrature methods for some logarithmic kernel integral equations on open arcs

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7th January 2002

Abstract

This paper is devoted to the solution of the Dirichlet problem for the Laplace and Helmholtz equation in the complement of a smooth open curve in the plane. The solution is looked for as a single layer potential, being therefore the corresponding density the solution of an integral equation on the open arc. This equation is transformed into an equivalent 1–periodic integral equation having existence and uniqueness of solution for any periodic data. Here we study the use of collocation and quadrature methods for solving this equation. We show the convergence of both methods and prove the existence of an asymptotic expansion of the error in powers of the discretization parameter. As consequence, we derive that for some of them a superconvergence phenomenon occurs when computing the solution of the differential problem. Two numerical experiments are shown in order to illustrate the theoretical results exposed in this work.

AMS Subject classification. 65R20

Key Words. Boundary integral equations, quadrature, collocation, superconvergence.

1 Introduction

Let Γ be a smooth parameterizable open arc in $\mathbb{R}^2$. In the domain defined by the complement of the curve, we state the Dirichlet problem for the homogeneous Laplace or Helmholtz equation. The solution is looked for as a single layer potential on the curve and consequently, the corresponding density must be a solution of an integral equation

*Research supported by MCYT/FEDER Project BFM2001-2521 and "Proyecto de Investigación del Gobierno de Navarra", Ref. 134/2002.

whose kernel contains the fundamental solution of the differential equation. Therefore, the kernel presents a logarithmic singularity on the diagonal.

The study of such equations can be made by means of the well-known cosine change of variable [19] which transforms the original integral equation into another one where the kernel, data and solution are even and 1–periodic. Moreover, the natural singularity which presents the density in the extremes of the arc is absorbed in the change. Hence, if the data is smooth, so is the solution of the new equation. This equation is solvable only for even data, and we must demand the solution to be also even to assure uniqueness. Consequently, the numerical methods have to take into account the parity of the new problem. For instance, Petrov–Galerkin methods must have even test and trial spaces (cf. [13, 11] and references therein). This restriction excludes a large class of numerical methods designed for solving periodic integral equations with logarithmic kernel.

Instead of this approach, we propose to modify the equation and construct a new equation of the same type such that the solution for even data coincides with that provided by the even periodic equation. This equation is constructed by adding and subtracting two different terms cancelling each other when applied to even functions. The resulting integral equation, say $A\tilde{A} = f$, has a smooth 1–periodic kernel except on the diagonal, where again a logarithmic singularity holds. For this equation, we can derive existence and uniqueness of solution for any f. Thus, we ignore the parity of the first equation in order to widen the numerical methods we can use. The first consequence is that, roughly speaking, the dimension of the problem is doubled. However, we will show in this work that there are some advantages which make this approach worthwhile.

In this paper we restrict ourselves to two families of methods: collocation and quadrature methods [12, 14]. Both of them depend on a parameter $''$ taken in $(-1{=}2; 1{=}2]$. Moreover, these methods can be formulated as Petrov–Galerkin methods making possible to carry out a global analysis for both of them. We will prove that there exists an expansion of the error in powers of h, the discretization parameter, similar to the results presented in [6, 8, 9]. Each power of h is multiplied by a function of $''$ and therefore, by choosing appropriately this parameter, we can drop a term from the expansion. Hence, gaining order can be seen as the cancellation of some terms in the asymptotic expansion. Here, we will show that an appropriate choice of this parameter allows to obtain numerical methods so that the potentials computed with the numerical densities converge with higher order than the convergence rates which the densities themselves are approximated. Moreover, the expansion of the error proves the optimality of the error bounds.

The functional frame where we set our problem is that of periodic Sobolev pseudodifferential operators. Stability and convergence of the methods are studied in the norms of these spaces. In this way, the stability of collocation methods in L^2 norm is formulated as an infimum-supremum condition. For the quadrature method a bound of the numerical solution in terms of the exact solution involving different norms for each of them suffices for stating a different inf-sup condition. On the other hand, an expansion in powers of h can be derived for the error committed when approximating a function by an optimal projection on the trial space under the action of the integral operator A. By using the inf-sup condition, we transform this expansion into an expansion of the error between the numerical solution and the optimal projection of the exact solution on the trial space. As byproduct, an expansion of the error between the exact and approximated solution of the

Laplace and Helmholtz equation holds. By examining carefully the terms of this expansion we deduce how ε can be taken in order to improve the accuracy of the approximation of the differential problem.

This paper is structured as follows. In the first section we set out the differential problem, the integral equation which determines the density, and the equivalent periodic equation we have to solve. The quadrature and collocation methods are also presented there. The properties of the integral equation are examined in the next section, proving existence and uniqueness of solution and the equivalence of the new equation with the original. In the following section, we show the convergence of the numerical schemes and prove the existence of an asymptotic expansion of the error. Afterwards, an expansion of the error for the computed solution of the differential problem is derived as a direct consequence. Moreover, the form of this expansion is explored, establishing the superconvergence of the potentials for particular values of ε. Finally, two numerical experiments are shown in order to illustrate the theoretical results.

2 The differential problem and numerical methods

Let Γ be an open curve in $\mathbb{R}^2$ given by a smooth, regular parameterization $\mathbf{x} : [-1, 1] \to \Gamma$. For $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda \geq 0$, we look for a solution of equation

$$\left| \begin{array}{ll} \Delta \omega + \lambda^2 \omega = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma, \\ \omega|_\Gamma = g, & \end{array} \right. \tag{1}$$

which can be expressed as a single layer potential, that is,

$$\omega(\mathbf{z}) = \int_{-1}^{1} \phi_\lambda(\mathbf{z} - \mathbf{x}(t))\, \varphi(t)\, \mathrm{d}t, \qquad \mathbf{z} \in \mathbb{R}^2 \setminus \Gamma. \tag{2}$$

Here ϕ_λ stands for the fundamental solution of $\Delta + \lambda^2$, namely

$$\phi_\lambda := \begin{cases} -\dfrac{1}{2\pi} \log(|\cdot|), & \text{if } \lambda = 0, \\ \dfrac{\mathrm{i}}{4} H_0^{(1)}(\lambda |\cdot|), & \text{if } \lambda \neq 0, \end{cases}$$

$H_0^{(1)}$ being the Hankel function of the first kind and order 0 [7, 11]. The density φ [11] must be a solution of

$$\int_{-1}^{1} \phi_\lambda(\mathbf{x}(s) - \mathbf{x}(t))\, \varphi(t)\, \mathrm{d}t = g(\mathbf{x}(s)), \quad s \in [-1, 1]. \tag{3}$$

If $\lambda \neq 0$, equation (3) is uniquely solvable (see [11] Chapter 11 and references therein). Actually, (2) is the unique solution of the Helmholtz equation satisfying the Sommerfeld conditions at infinity. For $\lambda = 0$, we have existence and uniqueness of solution for (3) if and only if the transfinite diameter, or logarithmic capacity, of Γ differs from 1 [16, 19]. Thus, we assume henceforth that the logarithmic capacity of Γ is different of 1 for $\lambda = 0$.

Notice that

$$H_0^{(1)}(x) = \tfrac{2\mathrm{i}}{\pi} I_0(-\mathrm{i}x)\log|x| + B(x), \tag{4}$$

where I_0 is the modified Bessel function of the first kind and B a smooth function. Since I_0 is also smooth, with $I_0(0) = 1$, the integral equations we have to solve have as kernels, smooth functions except on the diagonal, where a logarithmic singularity holds.

A way to study this equation is by means of the change of variable [19]

$$s = \cos(2\pi x), \quad t = \cos(2\pi y), \qquad x, y \in [0, 1/2].$$

Hence, defining

$$\begin{aligned} V(x,y) &:= \Phi_\lambda(\mathbf{x}(\cos(2\pi x)) - \mathbf{x}(\cos(2\pi y))), \\ \psi(y) &:= \pi\varphi(\cos(2\pi y))|\sin(2\pi y)|, \quad f(x) = g(\mathbf{x}(\cos(2\pi x))) \end{aligned} \tag{5}$$

and using the parity of ψ, f and V we obtain

$$\mathcal{V}\psi := \int_0^1 V(\cdot, y)\psi(y)\,\mathrm{d}y = f. \tag{6}$$

Once the density has been computed, the solution of (1) is recovered by

$$\int_0^1 \Phi_\lambda(\mathbf{z} - \mathbf{x}(\cos(2\pi y)))\psi(y)\,\mathrm{d}y, \qquad \mathbf{z} \in \mathbb{R}^2 \setminus \Gamma. \tag{7}$$

Notice that, for fixed $\mathbf{z}$, the function in the integral above is even, smooth and periodic.

In this paper, we propose a different approach. Notice that from the decomposition (4), we deduce that for $\lambda \neq 0$

$$V(x,y) := C(x,y)\log(|\cos(2\pi x) - \cos(2\pi y)|) + D(x,y), \tag{8}$$

where C, D are smooth, even and 1–periodic functions. Moreover, $C(x,x) \equiv -1/2\pi$. For $\lambda = 0$, the decomposition (8) holds trivially with $C \equiv -1/2\pi$. Define now

$$\begin{aligned} \Lambda\psi &:= -\frac{1}{2\pi}\int_0^1 \log(2\sin^2(\pi(\,\cdot - y)))\psi(y)\,\mathrm{d}y, \\ \widetilde{\Lambda}\psi &:= -\frac{1}{2\pi}\int_0^1 \log|\cos(2\pi\,\cdot) - \cos(2\pi y)|\psi(y)\,\mathrm{d}y, \end{aligned}$$

and

$$\mathcal{K} := \mathcal{V} - \widetilde{\Lambda}, \qquad \mathcal{A} := \Lambda + \mathcal{K}. \tag{9}$$

That is,

$$\mathcal{A}\psi = \int_0^1 A(\,\cdot\,, y)\psi(y)\,\mathrm{d}y$$

being

$$A(x,y) := -\frac{1}{2\pi}\log(2\sin^2(\pi(x-y))) + \Big(V(x,y) + \frac{1}{2\pi}\log|\cos(2\pi x) - \cos(2\pi y)|\Big)$$

We remark that the second addendum above is a continuous function. Now, instead of solving (6), we work with equation

$$\mathrm{A}\varphi = f, \tag{10}$$

with f given in (5). In the next section, we will show that any solution of (10) is even and solves (6). Hence, (10) can be used for computing φ and consequently, the solution of (1).

The numerical methods we will work on are quadrature and collocation methods. Both of them depend on a parameter $\varepsilon \in (-1/2, 1/2]$. Given N a positive integer, define $h := 1/N$ and $x_\alpha := \alpha h$ where $\alpha \in \mathbb{Z}$ or $\alpha \in \varepsilon + \mathbb{Z}$. Then, the quadrature method is defined by

$$\left|\begin{array}{l} (\varphi_1, \ldots, \varphi_N) \in \mathbb{C}^N \\ h \displaystyle\sum_{j=1}^{N} A(x_{i+\varepsilon}, x_j)\varphi_j = f(x_{i+\varepsilon}), \qquad i = 1, \ldots, N. \end{array}\right.$$

Notice ε cannot be taken equal to 0 since this choice involves evaluating on the diagonal of the kernel. This method do not require the use of numerical integration and therefore is directly implementable.

To define the collocation method, we have to introduce first the space of smoothest periodic splines of degree $d \geq 1$ given by

$$S_h^d := \left\{\varphi_h \in \mathcal{C}^{d-1}(\mathbb{R}) \mid \varphi_h = \varphi_h(1 + \cdot), \quad \varphi_h|_{[x_i, x_{i+1}]} \in \mathbb{P}^d, \ \forall i \in \mathbb{Z}\right\}.$$

Here $\mathbb{P}^d$ denotes the set of polynomials of degree at most d. For $d = 0$, S_h^d is simply the space of 1–periodic piecewise constant functions over the grid $\{x_i\}$. The collocation method is then defined by the scheme

$$\left|\begin{array}{l} \varphi_h \in S_h^d, \\ \displaystyle\int_0^1 A(x_{i+\varepsilon}, y)\varphi_h(y)\,\mathrm{d}y = f(x_{i+\varepsilon}), \qquad i = 1, \ldots, N. \end{array}\right.$$

3 Functional settings

The natural frame to study equation (6) is that given by the periodic Sobolev spaces. Let $\mathcal{D}$ be the metric space of the smooth 1–periodic functions and denote with $\mathcal{D}'$ its dual space. For $u \in \mathcal{D}'$ the Fourier coefficient $\widehat{u}(m)$ is defined simply as the dual product of u by $\exp(-2\pi \mathrm{i} m x)$. Clearly, it is an extension of the classical Fourier coefficients for integrable functions.

The 1–periodic Sobolev space of order $s \in \mathbb{R}$ is then defined by

$$H^s := \left\{\varphi \in \mathcal{D}' \,\middle|\, \|\varphi\|_s < \infty\right\}, \qquad \|\varphi\|_s^2 := |\widehat{\varphi}(0)|^2 + \sum_{m \neq 0} |m|^{2s} |\widehat{\varphi}(m)|^2.$$

It is well known that H^s is a Hilbert space and that the inclusion of H^s in H^t for $s > t$ is compact. For a non-negative integer m, H^m is just the space of 1–periodic functions with the mth derivative locally square-integrable endowed with the usual Sobolev norm.

Moreover, $\mathcal{D}$ is dense in H^s for all $s \in \mathbb{R}$. For a review of these properties, see for instance [11].

We will work also with the even/odd Sobolev spaces H^s_e and H^s_o defined by

$$H^s_e \;:=\; \left\{\tilde{A} \in H^s \mid \hat{\tilde{A}}(m) = \hat{\tilde{A}}(-m); \quad \forall m \in \mathbb{Z}\right\};$$
$$H^s_o \;:=\; \left\{\tilde{A} \in H^s \mid \hat{\tilde{A}}(m) = -\hat{\tilde{A}}(-m); \quad \forall m \in \mathbb{Z}\right\}:$$

Clearly, $H^s = H^s_e \oplus H^s_o$. From now on, we will say that $\tilde{A}$ is even or odd if it belongs to H^s_e or H^s_o respectively.

On the other hand, $\mathsf{B} : \mathcal{D}' \to \mathcal{D}'$ is a pseudodifferential operator ($\tilde{A}$do in short) of order n if $\mathsf{B} : H^s \to H^{s-n}$ is continuous for all $s \in \mathbb{R}$. Following [11], a $\tilde{A}$do B of order n which does not change the parity of the functions, that is, with $\mathsf{B} : H^s_e \to H^{s-n}_e$ and $\mathsf{B} : H^s_o \to H^{s-n}_o$ well-defined, is said to be an even $\tilde{A}$do.

Proposition 3.1 V *is an even* $\tilde{A}$*do of order* -1*. Moreover,* $\mathsf{V}|_{H^s_o} \equiv 0$ *and* $\mathsf{V} : H^s_e \to H^{s+1}_e$ *is an isomorphism.*

Proof. Consider V the kernel of V. Since V is even in both variables, we deduce $\mathsf{V}|_{H^s_o} \equiv 0$. On the other hand, from decomposition (8) we derive the continuity of $\mathsf{V} : H^s_e \to H^{s+1}_e$ for all $s \in \mathbb{R}$ ([11] Chapter 11). Moreover, V is Fredholm of index zero. Since equation (3) has a unique solution, the unique even solution of $\mathsf{V}\tilde{A} = 0$ is the trivial one. Therefore, $\mathsf{V} : H^s_e \to H^{s+1}_e$ is invertible. ¤

By this Proposition, $\mathsf{V}\tilde{A} = f$ is solvable only if f is even and the uniqueness of the solution is attained by requiring the solution to be also even. Therefore, we conclude that for solving numerically we must use numerical schemes in H^s_e.

The situation is different for A. Since [11, 19]

$$\Lambda\tilde{A} := \frac{1}{2¼}\log(4)\hat{\tilde{A}}(0) + \frac{1}{2¼}\sum_{m\neq 0}\frac{1}{|m|}\hat{\tilde{A}}(m)\exp(2¼\mathrm{i}m\,\cdot);$$

$\Lambda : H^s \to H^{s+1}$ is an even $\tilde{A}$do operator with continuous inverse being its restrictions to H^s_o and H^s_e also invertible.

Proposition 3.2 A *is an even invertible* $\tilde{A}$*do of order* -1*. Moreover,* $\mathsf{A}|_{H^s_e} = \mathsf{V}|_{H^s_e}$ *and* $\mathsf{A}|_{H^s_o} = \Lambda|_{H^s_o}$*.*

Proof. By a change of variable, one can check that for any $\tilde{A}$ even

$$\begin{aligned}\widetilde{\Lambda}\tilde{A} &= -\frac{1}{2¼}\int_0^1 \log|2\sin(¼(\cdot - y))|\tilde{A}(y)\,\mathrm{d}y - \frac{1}{2¼}\int_0^1 \log|\sin(¼(\cdot + y))|\tilde{A}(y)\,\mathrm{d}y \\ &= -\frac{1}{2¼}\int_0^1 \log(2\sin^2(¼(\cdot - y)))\tilde{A}(y)\,\mathrm{d}y = \Lambda\tilde{A} \end{aligned} \qquad (11)$$

and therefore $\widetilde{\Lambda}|_{H^s_e} = \Lambda|_{H^s_e}$. On the other hand $\widetilde{\Lambda}|_{H^s_o} \equiv 0$. Therefore, for $\tilde{A} = \tilde{A}_e + \tilde{A}_o$ with $\tilde{A}_e \in H^s_e$ and $\tilde{A}_o \in H^s_o$, it yields

$$\mathsf{A}\tilde{A} = \Lambda\tilde{A}_0 + \mathsf{V}\tilde{A}_e$$

from where the result follows readily. ¤

Hence, (10) has a unique solution for any f, even or not, and for even data, provides the unique even solution of equation (6).

Lemma 3.3 *The operator* K *defined in* (9) *is an even* Ã*do operator of order* -3.

Proof. If ¸ $= 0$, it follows easily that the kernel of K is smooth and 1–periodic. Then it defines actually an operator of order $-\infty$, that is, continuous from H^s into H^t for all $s; t \in \mathbb{R}$ [10, 11].

Suppose now that ¸ $\neq 0$. Since the kernel is even, we have $\mathsf{K}|_{H^s_o} \equiv 0$. Thus we can restrict ourselves to studying $\mathsf{K}|_{H^s_e}$. We remark that $I_0'(0) = 0$ and $I_0''(0) \neq 0$. Hence, proceeding as in (11),

$$\mathsf{K}\tilde{A} := \int_0^1 E(\cdot; y) \sin^2(¼(\cdot - y)) \log(\sin^2(¼(\cdot - y)))\tilde{A}(y)\,\mathrm{d}y + \int_0^1 F(\cdot; y)\tilde{A}(y)\,\mathrm{d}y$$

for any $\tilde{A}$ even with $E; F$ smooth and periodic. Now it is well now that such an operator has order -3 [8, 11]. ¤

4 Convergence properties and an error expansion

Let

$$S_h^{-1} := \Big\{ \sum_{j=1}^{N} {}^1{}_j ±_{x_j} \,\Big|\, {}^1{}_j \in \mathbb{C};\ j = 1; : : : ; N \Big\}; \quad T_h^{\varepsilon} := \Big\{ \sum_{j=1}^{N} {}^1{}_j ±_{x_{j+\varepsilon}} \,\Big|\, {}^1{}_j \in \mathbb{C};\ j = 1; : : : ; N \Big\};$$

with $±_z$ denoting the Dirac delta at point z. Notice that $S_h^{-1}; T_h^{\varepsilon} \subset H^s$ for $s < -1=2$. We introduce the notation

$$\langle u; ±_z \rangle := u(z):$$

The bracket $\langle \cdot ; \cdot \rangle$ is not necessarily a duality product for continuous function. We admit that the first term has discontinuities but not on z. Collocation and quadrature methods can be formulated as

$$\left|\begin{array}{l} \tilde{A}_h \in S_h^d; \\ \langle \mathsf{A}\tilde{A}_h; r_h \rangle = \langle f; r_h \rangle; \qquad \forall r_h \in T_h^{\varepsilon} \end{array}\right.$$

being therefore Petrov–Galerkin methods with S_h^d and T_h^{ε} as trial and test spaces.

When studying the convergence rates of the methods, the roots of the functions

$$C_k := \sum_{m=1}^{\infty} \frac{1}{m^k} \exp(2¼\mathrm{i}m\,\cdot\,) - \sum_{m=-\infty}^{-1} \frac{1}{m^k} \exp(2¼\mathrm{i}m\,\cdot\,);$$

defined for any positive integer k, play an essential role. These functions have been profusely treated in the literature [3, 11, 17]. Obviously, C_k is a 1–periodic function, odd for k even and vice versa. If k is even, C_k have only two roots in the interval $(-1=2; 1=2]$, namely $\{0; 1=2\}$. For k odd there exist only two roots in $(-1=2; 1=2)$, symmetrically placed with respect to the origin. From now on, and for any k odd, we will denote by ´(k) the unique root of C_k in $(0; 1=2)$. In case $k = 1$, $C_1 = -\log(4\sin^2(¼\cdot))$, and therefore ´$(1) = 1=6$.

Proposition 4.1 *The collocation and quadrature methods are convergent if $\varepsilon \neq 0$ for d even, $\varepsilon \neq 1/2$ for $d \geq 1$ odd and $\varepsilon \neq 0, 1/2$ for $d = -1$. Furthermore for all $-1 \leq s \leq t \leq d+1$, with $t > -1/2$ and $s < d+1/2$ there exists C independent of ψ and h such that*

$$\|\psi - \psi_h\|_s \leq C h^{t-s} \|\psi\|_t. \tag{12}$$

Besides, if d is even and $\varepsilon = 1/2$ or d is odd and $\varepsilon = \pm\eta(d+2)$, it holds

$$\|\psi - \psi_h\|_{-2} \leq C h^{d+3} \|\psi\|_{d+2}.$$

Proof. This result is well known for equation $\Lambda\psi = f$ (cf. [12, 14, 15, 17] and references therein). Smoother perturbations can be treated using standard techniques. See for instance [11] Theorems 13.5.2 and 13.5.3. ∎

Notice that as byproduct we deduce that for $d \geq 0$,

$$\|\psi_h\|_0 \leq C \|\psi\|_0 \tag{13}$$

with C independent of h and ψ. That is, the collocation is stable in H^0. For $d = -1$, i.e. for the quadrature method, the bound

$$\|\psi_h\|_{-1} \leq C \|\psi\|_0, \tag{14}$$

derived straightforwardly from the convergence estimate, will be enough for our purposes.

Proposition 4.2 *For $d \geq 0$, and h small enough, there exists $\beta > 0$ independent of h such that*

$$\inf_{\psi_h \in S_h^d} \sup_{r_h \in T_h^\varepsilon} \frac{|\langle A\psi_h, r_h\rangle|}{\|\psi_h\|_0 \|r_h\|_{-1}} \geq \beta$$

whereas for $d = -1$ and h sufficiently small, there exists $\beta > 0$ such that

$$\inf_{\psi_h \in S_h^{-1}} \sup_{r_h \in T_h^\varepsilon} \frac{|\langle A\psi_h, r_h\rangle|}{\|\psi_h\|_{-1} \|r_h\|_{-1}} \geq \beta.$$

Proof. If $d \geq 0$, this is a consequence of (13) (cf. [5]). For $d = -1$, the proof can be adapted from the previous case by using now (14). See also [6] for a different proof. ∎

In the rest of this section we will derive an expansion of the error in powers of the discretization parameter h. As in [6, 8] we will use an optimal projection on S_h^d of the exact solution to compare with the numerical solution. This projection is constructed by matching the central Fourier coefficients

$$\psi \quad \longmapsto \quad S_h^d \ni D_h^d \psi \quad \text{s.t.} \quad \widehat{D_h^d \psi}(\mu) = \widehat{\psi}(\mu), \qquad -N/2 < \mu \leq N/2.$$

This map is well defined and satisfies the convergence estimate

$$\|D_h^d \psi - \psi\|_s \leq C h^{t-s} \|\psi\|_t$$

for $s \leq t \leq d+1$, $s < d+1/2$ [6, 8]. Actually, $D_h^d \psi$ converges to ψ in all available Sobolev norms with the same order as the best approximation of ψ in S_h^d [10].

Theorem 4.3 *There exists a sequence of Ãdo* T_k *of order* $k-1$ *independent of* $\tilde{A}$*,* h *and* '' *such that*

$$\langle \mathsf{A}(D_h^d\tilde{A}-\tilde{A}); r_h\rangle = \sum_{k=d+2}^{M} h^k\, C_k('')\langle \mathsf{T}_k\tilde{A}; r_h\rangle + \mathcal{O}(h^{M+1})\|\tilde{A}\|_{M+1}\|r_h\|_{-1}:$$

Proof. Notice that if $\tilde{A}$ is even/odd so is $D_h^d\tilde{A}$. Hence, for $\tilde{A} \in H^{M+1}$, and taking $\tilde{A}_e \in H_e^{M+1}$ and $\tilde{A}_o \in H_o^{M+1}$ such that $\tilde{A}_e + \tilde{A}_o = \tilde{A}$, the following decomposition holds

$$\mathsf{A}(D_h^d\tilde{A}-\tilde{A}) = \Lambda(D_h^d\tilde{A}_e-\tilde{A}_e) + \mathsf{V}(D_h^d\tilde{A}_o-\tilde{A}_o):$$

From (8), and proceeding as in (11), it follows

$$\begin{aligned} \mathsf{V}\tilde{A}_e &= \int_0^1 \acute{A}_\lambda(\mathbf{x}(\cos(2¼\cdot)-\mathbf{x}(\cos(2¼y)))\tilde{A}_e(y)\,\mathrm{d}y \\ &= \int_0^1 C(\cdot;y)\log(2\sin^2(¼(\cdot-y)))\tilde{A}_e(y)\,\mathrm{d}y + \int_0^1 D(\cdot;y)\tilde{A}_e(y)\,\mathrm{d}y; \end{aligned}$$

C and D being smooth 1–periodic functions. Therefore, in both cases we are applying a integral operator with logarithmic kernel to the error committed by the projection D_h^d. The result is consequence of applying [8] Corollary 8 and Remark 9 for $d \geq 0$ or [6] Theorem 7 for $d=-1$, where this kind of expressions are analyzed. ¤

From now on, we will make relevant the parameter '' in the numerical solution. Hence, we will denote by $\tilde{A}_h^\varepsilon$ the solution given by the numerical scheme. Consider also

$$m = m('';d) := \begin{cases} d+3; & \text{if } '' = \pm´(d+2) \text{ and } d \text{ odd or } '' = 1{=}2 \text{ and } d \text{ even;} \\ d+2; & \text{otherwise:} \end{cases}$$

Theorem 4.4 *There exists a sequence of Ãdo* R_k^ε *of order* k *such that*

$$\left\| D_h^d\tilde{A} - \tilde{A}_h^\varepsilon - \sum_{k=m}^{M} h^k D_h^d \mathsf{R}_k^\varepsilon \tilde{A}\right\|_\tau \leq Ch^{M+1}\|\tilde{A}\|_{M+1};$$

being ¿ $=-1$ *if* $d=-1$ *and* ¿ $=0$ *otherwise, and* C *independent of* h *and* $\tilde{A}$*. Moreover,* $\mathsf{R}_j^\varepsilon = (-1)^{j+1}\mathsf{R}_j^{-\varepsilon}$ *for* $j = m; : : : ; 2m-1$.

Proof. We remark that

$$\left|\left\langle \mathsf{A}\Big(D_h^d\tilde{A} - \tilde{A} - \sum_{k=m}^{M} h^k C_k('')\mathsf{A}^{-1}\mathsf{T}_k\tilde{A}\Big); r_h\right\rangle\right| \leq Ch^{M+1}\|\tilde{A}\|_{M+1}\|r_h\|_{-1}:$$

Let Q_h^ε be the map which for any $\tilde{A}$ solution of $\mathsf{A}\tilde{A} = f$ assigns $\tilde{A}_h^\varepsilon$, the numerical solution given by the numerical scheme. Then Theorem 4.3 and the inf-sup condition at Proposition 4.2 prove that

$$\left\| D_h^d\tilde{A} - \tilde{A}_h^\varepsilon - \sum_{k=m}^{M} h^k C_k('')Q_h^\varepsilon \mathsf{A}^{-1}\mathsf{T}_k\tilde{A}\right\|_\tau \leq Ch^{M+1}\|\tilde{A}\|_{M+1}:$$

The result follows readily by induction. Since $\mathsf{R}_j^{\varepsilon} = C_j('')\mathsf{A}^{-1}\mathsf{T}_j$ for $j = m; : : : ; 2m - 1$, the symmetry of the first terms of the expansion with respect to the parameter $''$ is consequence of the parity properties of C_m. ¤

Notice that from the approximation properties of D_h^d we deduce

$$\|\tilde{A} - \tilde{A}_h^{\varepsilon}\|_{d-m+1} \leq Ch^m \|\tilde{A}\|_m;$$

that is, we recover the order of convergence given in Proposition 4.1 but with the assumption of a somewhat higher regularity. This extra order of smoothness is a consequence of the asymptotic analysis used here.

5 Superconvergence

In this section our aim is to study the error committed when approximating (1) using the density computed by the numerical schemes. Firstly we prove that an expansion of the error is inherited. The potential is computed by applying an integral operator whose kernel is smooth and even in the variable of integration. This gives rise to some peculiarities in the expansion which are exploited in order to obtain a superconvergence for the potentials.

The proofs are based upon parity properties of the solution and of the terms appearing in the expansion with respect to the parameter $''$. In order to treat both the quadrature and collocation method, for $r_h \in S_h^{-1}$ given by

$$r_h = \sum_{j=1}^{N} r_j \pm_{x_j}$$

we will denote

$$r_h(-\cdot) := \sum_{j=1}^{N} r_j \pm_{-x_j}:$$

Lemma 5.1 *Let $\tilde{A}_h^{\varepsilon}$, $\tilde{A}_h^{-\varepsilon}$ be the numerical solutions for $\mathsf{A}\tilde{A} = f$ with f even. Then*

$$\tilde{A}_h^{-\varepsilon} = \tilde{A}_h^{\varepsilon}(-\cdot):$$

Proof. We prove the result for $d \geq 0$. Let $('_j)_{j=1}^N$ be the B-spline basis of S_h^d [17]. Then, it holds

$$'_j(\cdot - x_k) = '_{j+k};$$

where we have identified $'_{k+mN} \equiv '_k$ for all $k = 1; : : : ; N$, $m \in \mathbb{Z}$. Then if

$$\tilde{A}_h^{-\varepsilon} := \sum_{j=1}^{N} ¸_j^{-\varepsilon} '_j$$

is the numerical solution, it holds that

$$\sum_{j=1}^{N} ¸_j^{-\varepsilon} \mathsf{A}\, '_j(x_{N-i-\varepsilon}) = \sum_{j=1}^{N} ¸_j^{-\varepsilon} (\Lambda\, '_j(x_{N-i-\varepsilon}) + \mathsf{K}\, '_j(x_{N-i-\varepsilon})) = f(x_{N-i-\varepsilon}): \quad (15)$$

Since $'_j(-\,\cdot\,) = '_{N-j}$, it follows

$$\begin{aligned}\Lambda\, '_j(x_{N-i-\varepsilon}) &= -\frac{1}{2¼}\int_0^1 \log(2\sin^2(¼(x_{N-i-\varepsilon}-y)))\, '_j(y)\,\mathrm{d}y \\ &= -\frac{1}{2¼}\int_0^1 \log(2\sin^2(¼(x_{i+\varepsilon}-y)))\, '_{N-j}(y)\,\mathrm{d}y = \Lambda\, '_{N-j}(x_{i+\varepsilon}). \qquad (16)\end{aligned}$$

Analogously, it can be checked that

$$\mathrm{K}\, '_j(x_{N-i-\varepsilon}) = \mathrm{K}\, '_{N-j}(x_{N-i-\varepsilon}) = \mathrm{K}\, '_{N-j}(x_{i+\varepsilon}). \qquad (17)$$

Hence, applying (16) and (17) in (15), we obtain

$$¸_N^{-\varepsilon}\mathrm{A}\, '_N(x_{i+\varepsilon}) + \sum_{j=1}^{N-1} ¸_j^{-\varepsilon}\mathrm{A}\, '_{N-j}(x_{i+\varepsilon}) = f(x_{N-i-\varepsilon}) = f(x_{i+\varepsilon}) = \sum_{j=1}^{N} ¸_j^{\varepsilon}\mathrm{A}\, '_j(x_{i+\varepsilon});$$

and therefore $¸_{N-j}^{-\varepsilon} = ¸_j^{\varepsilon}$, $j = 1;:::;N-1$ and $¸_N^{-\varepsilon} = ¸_N^{\varepsilon}$ from where the result follows readily. The result for the quadrature method can be proved in a similar way. ¤

Let X be a normed space with norm $\|\cdot\|_X$ and $W : H^s \to X$ be a continuous operator for all $s \in \mathbb{R}$ satisfying

$$W(Ã(-\,\cdot\,)) = WÃ. \qquad (18)$$

The last result of this section shows how the expansion of the error is under the action of such an operator. To formulate this result, we introduce the quantity $n = n(";d)$, given in Table 1.

d	odd	odd	0	0	≥ 2 even	≥ 2 even
$"$	$\pm´(d+2)$	$\neq \pm´(d+2)$	$\pm´(3)$	$\neq \pm´(3)$	$\pm´(d+3)$	$\neq \pm´(d+3)$
n	$d+4$	$d+2$	4	3	$d+5$	$d+3$

Table 1: Values of n in function of d and $"$

Theorem 5.2 *For all $Ã \in H_e^{M+1}$ there exists C independent of h and $Ã$ such that*

$$\Big\| WÃ - WÃ_h^{\varepsilon} - \sum_{k=n}^{M} h^k W\mathrm{R}_k^{\varepsilon}Ã \Big\|_X \leq Ch^{M+1}\|Ã\|_{M+1}.$$

Proof. Since $W : H^s \to X$ is continuous for all $s \in \mathbb{R}$, we derive from Theorem 4.4

$$\Big\| WÃ - WÃ_h^{\varepsilon} - \sum_{k=m}^{M} h^k W\mathrm{R}_k^{\varepsilon}Ã \Big\|_X \leq Ch^{M+1}\|Ã\|_{M+1}.$$

Property (18) and Lemma 5.1 yield $WÃ_h^{\varepsilon} = WÃ_h^{-\varepsilon}$ and therefore

$$W\mathrm{R}_j^{\varepsilon}Ã = W\mathrm{R}_j^{-\varepsilon}Ã$$

for all j. On the other hand, we have that $\mathrm{R}_j^{\varepsilon} = (-1)^{j+1}\mathrm{R}_j^{-\varepsilon}$, for $j = m+1; : : : ; 2m-1$. Consequently, the expansion contains no even powers of h in the range $\{m; : : : ; 2m-1\}$. The result is a straightforward consequence of examining each case carefully. ¤

Notice that as a straightforward consequence we have the following result.

Theorem 5.3 *There exists* $C > 0$ *independent of* h *such that for all* $\tilde{A} \in H^n$

$$\|W\tilde{A} - W\tilde{A}_h^{\varepsilon}\|_X \leq Ch^n\|\tilde{A}\|_n:$$

Remark. For $\mathbf{z} \in \mathbb{R}^2 \setminus \Gamma$ and $d \geq 0$,

$$!_h(\mathbf{z}) = W\tilde{A}_h^{\varepsilon}(\mathbf{z}) := \int_0^1 \acute{A}_\lambda(\mathbf{z} - \mathbf{x}(\cos(2¼x)))\tilde{A}_h^{\varepsilon}(x)\,\mathrm{d}x;$$

whereas for $d = -1$

$$!_h(\mathbf{z}) = W\tilde{A}_h^{\varepsilon}(\mathbf{z}) := \sum_{j=1}^{N} \tilde{A}_j \acute{A}_\lambda(\mathbf{z} - \mathbf{x}(\cos(2¼x_j)))$$

where $\tilde{A}_h^{\varepsilon} = \sum_{j=1}^{N} \tilde{A}_j \pm_{x_j} \in S_h^{-1}$ is the solution of the quadrature method. Obviously, $!_h$ is an approximation of the solution of (1) (see (7)). For K any compact set in $\mathbb{R}^2 \setminus \Gamma$, $W : H^s \to \mathcal{C}(K)$ is continuous for all $s \in \mathbb{R}$, satisfying in addition property (18). By Theorem 5.2, we deduce that

$$\max_{\mathbf{z}\in K} |!_h(\mathbf{z}) - !(\mathbf{z})| \leq Ch^n\|\tilde{A}\|_n:$$

Thus, taking the optimal " according to Table 1, we obtain approximations of the potential converging faster (in one or two additional orders) than the order which the density itself is being approximated by the numerical scheme. Moreover, the use of Richardson extrapolation is fully justified as a way of improving the accuracy of the solution, by defining a new set of solutions converging faster, or for obtaining a global posteriori error estimates.

6 Numerical experiments

In this section we expose two numerical experiments to illustrate the theoretical results shown in this work.

Helmholtz Equation

Let Γ be the arc of parabola given by $\mathbf{x}(t) := (t; t^2)$, $t \in [-1; 1]$ and f such that $f(\mathbf{x}(t)) = t^2$. We want to solve numerically the problem

$$\left|\begin{array}{l} \Delta ! + (1+\mathrm{i})^2 ! = 0 \\ !|_\Gamma = f \\ |!(\mathbf{z}) - \mathrm{i}(1+\mathrm{i})@_r !(\mathbf{z})| = o(|\mathbf{z}|^{-1/2}): \end{array}\right.$$

The last restriction is just the Sommerfeld condition in the plane. For solving the corresponding equation (10) we use the quadrature method with ″ = 1=6. Recall that with this choice the method has order 2 and the numerical potential converges with order 3. We have applied the method with N = 32; 48; 72; 108; 162 and 243 (notice that the rate between two consecutive grid steps is equal to 3=2) and computed the potential at 20 equidistant points placed on the circumference centered at origin with radius 2. In addition, we have applied one step of Richardson extrapolation. As exact solution of the integral equation to compare with we have considered that given by taken $N = 2000$ in the same method.

In Figure 1 we plot the logarithm of the error versus $-\log(N)$ for the potential, the circles, and the first step of extrapolation, the cross. The solid line represents order three whereas order four is depicted by the dashed line. We observe clearly that the estimate rates of converge correspond with the results predicted by the theory.

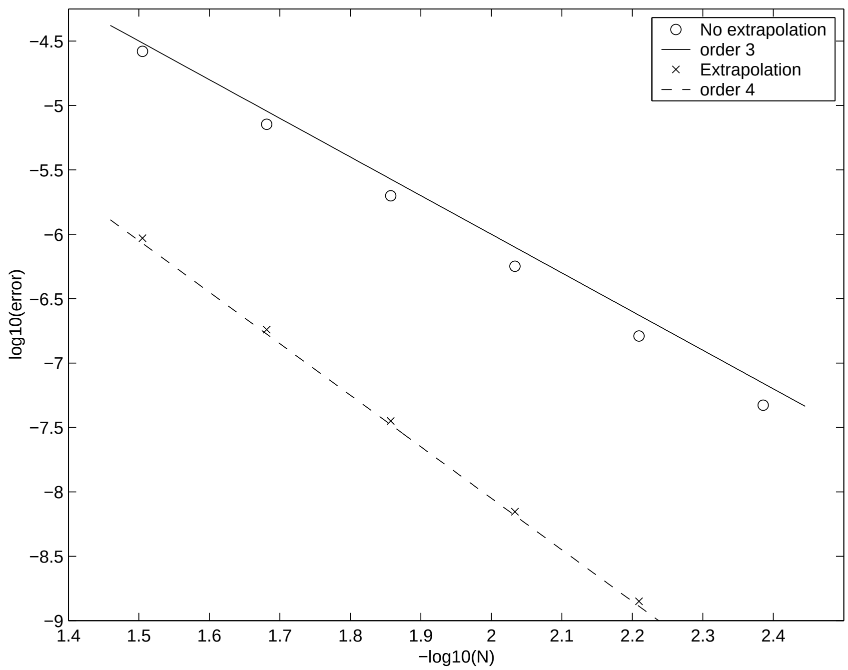

Figure 1: Numerical results for the Helmholtz equation

Laplace equation

Here we look for the solution of

$$\left| \begin{array}{ll} \Delta ! = 0; & \text{in } \mathbb{R}^2 \setminus \Gamma \\ !|_\Gamma = f & \end{array} \right.$$

which can be written as (2), being Γ the arc of circumference (cos(¼t=2); sin(¼t=2)) with $t \in [-1; 1]$ and $f(\mathbf{x}(t)) = (t^2 - 1)$. We apply the collocation method with piecewise constant functions as trial space and fix ″ = ´(3) ≈ 0:23082930516101 using the same step sizes as example before. This choice of ″ defines a method of order 3 with the potentials converging with order 4, according to Theorem 5.2 and Table 1. We remark that the method requires now the use of numerical integration which is carried out by using the ideas presented in [4].

The solution of Laplace equation is calculated at 20 equally spaced points on the segment $\{2\} \times [-1; 1]$. Again, one step of Richardson extrapolation has been applied to define a new set of solutions. The results are collected in Figure 2 where we have plotted the logarithm of the error versus $-\log(N)$ for both sets of solutions. Order four and five are represented with a solid and a dashed line respectively. We observe that the numerical experiments agree with the theoretical results.

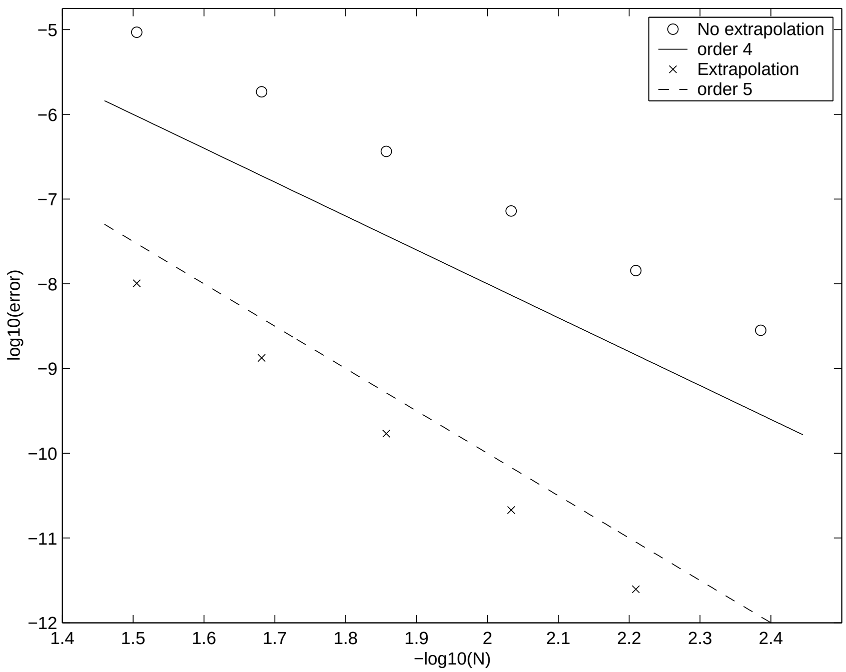

Figure 2: Numerical results for the Laplace equation

Acknowledgement. The author wants to thank Prof. F.-J. Sayas for the discussion which motivated this work and for the valuable comments which helped to improve both the content and presentation of this paper.

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