Local expansions of periodic spline interpolation with some applications

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Abstract. *In this paper we prove some asymptotic expansions of the error of interpolation on equally spaced nodes with periodic smoothest splines of arbitrary degree on a uniform partition. We obtain a local expansion in terms of derivatives of the interpolate. Afterwards we apply this result to the asymptotic study of the numerical solution of periodic integral equations of the second kind by means of −collocation methods. We show some new superconvergence results and give particular forms of these expansions depending on the choices of the parameter . We finally give some numerical experiments, which corroborate the theory.*

1 Introduction

In this paper we work on some asymptotic properties of periodic interpolation with splines. Periodic smoothest splines have been widely used as a means of obtaining good approximations for integral and pseudodifferential equations and have therefore been extensively studied in this context. A number of classical and recent results on periodic splines and their applications can be found in [9].

When a periodized B–spline basis is used, interpolation with periodic splines requires the solution a linear system with circulant–banded form. This system is in fact circulant if everything (i.e. the position of the knots and of the interpolation nodes) is uniform. However, the problem has a strong local behaviour in the sense that modifications in one part of the interpolated function modify the spline in nearby zones, whereas changes in distant areas are minor. Our study is centered in proving a formula that specifies this essentially local effect of interpolation, despite its global definition. We also show a simple application of our result to obtain asymptotic expansions of the error of collocation methods for integral equations. In its turn, these expansions justify the use of Richardson extrapolation for acceleration of convergence and for a posteriori error estimation.

The structure of the paper is as follows. In Section 2 we briefly introduce the problem and state (without all the hypotheses) the main result. Then we revise some functional spaces, based on the Fourier series of periodic functions, and write the interpolation as a linear system for the principal Fourier coefficients of the unknown. Section 3 is devoted to proving the local expansion of the interpolate. Let u_h be the interpolated of u in the space of periodic smoothest splines of degree d on a uniform grid of width $h = 1/N$, with equally spaced interpolation nodes. Then the result basically states that

$$u_h(x) - u(x) = h^{d+1}\underline{P}_{d+1}(x/h)u^{(d+1)}(x) + h^{d+2}\underline{P}_{d+2}(x/h)u^{(d+2)}(x) + \dots$$

being $\{\underline{P}_k\}$ the periodization of some polynomials. These polynomials depend on the degree of the splines and on the ratio of the difference between nodes (interpolation points) and knots (breakpoints) and the meshwidth. A great part of the work is taken in having some asymptotic expansions for a function Δ_d related to the Fourier form of the interpolation problem and which has already appeared in the literature of numerical methods for integral equations using this kind of discrete spaces (see [6] and [16]). We give explicit forms of the first group of polynomials P_k, which are related to the well–known Bernoulli polynomials.

A particular case of this result, namely the 'symmetric case' (midpoint interpolation if the degree is even and knot interpolation if it is odd), had already appeared in [13]. The expansion had been proved there by a careful use of Taylor expansions instead of Fourier techniques. However, the analysis does not give way to recognition of the explicit forms of the polynomials P_k and the extension to the general case is rather involved.

In Section 4 we apply the previous results to the study of the $\epsilon-$collocation method for periodic integral equations of the second kind with smooth kernel:

$$u + \int_0^1 K(\cdot, x)u(x)dx = f.$$

We show the effect of applying a smooth kernel integral operator to the interpolate and employ it to obtain asymptotic expansions of the error and nodal superconvergence results as a byproduct. If we apply a smooth integral operator to the solution, as happens with reconstruction formulae in boundary element methods or with the Sloan iteration of the solution, we obtain again an asymptotic expansion of the error. We show how optimal convergence is obtained for special choices of the collocation points and how the symmetric cases give rise to expansions in even powers of the discretization parameter. For instance, we prove that in the case of polygonals, two sets of collocation points give methods of order three, which to the best of our knowledge had not been previously noticed.

Results of this kind had been obtained in [7] for logarithmic kernel integral equations of the first kind with spline Galerkin methods. Some other results, with a much more general analysis referred to periodic pseudodifferential equations and the action of a smoothing functional upon the solution given by collocation methods, appear in [11]. We will compare them with ours in the same section.

Finally we give some numerical tests for several of the phenomena studied in Section 4, when applying a polygonal based $\epsilon-$collocation method to a boundary integral equation.

Let us finally mention that we omit the simplest case of piecewise constant functions for several reasons. First of all, interpolation with piecewise constants is a purely local method (the only other exception here is knot interpolation with polygonals) and therefore the results are trivial by Taylor expansions. On the other hand, the convergence of the Fourier series of a piecewise constant function is neither absolute nor uniform. Since most of our analysis is played with these rules, inclusion of this trivial case would greatly complicate notations and statements of the results.

Notation. In this work we extensively deal with $1-$periodic functions. They will be identified by their restriction to the interval $[0, 1)$. As a general rule, both $u : [0, 1) \to \mathbf{R}$ and its $1-$periodic extension to $\mathbf{R}$ will be denoted with the same symbol. However, we will make an exception of this for polynomials, always denoted by capital letters. Therefore, if P is a polynomial, $\underline{P}$ is given by

$$\underline{P}(x+n) := P(x), \quad \forall x \in [0,1), \quad \forall n \in \mathbf{Z}\setminus\{0\}.$$

Also, the symbol $\sum_{k\neq 0}$ is to be understood as summation over $k \in \mathbf{Z}, k \neq 0$.

2 The problem of periodic interpolation

In the sequel $\mathcal{C}^d$ will denote the set of 1−periodic functions whose dth derivative is continuous. Let N be a positive integer, $h := 1/N$ and let us consider the points

$$x_i := ih, \quad i \in \mathbf{Z}.$$

For $d \geq 1$, we define the space of 1−periodic smoothest splines of degree d with breakpoints at the grid $\{x_i \mid i \in \mathbf{Z}\}$,

$$S_h^d := \left\{u : \mathbf{R} \to \mathbf{R} \mid u \in \mathcal{C}^{d-1}, u|_{(x_i,x_{i+1})} \in \mathbf{P}_d\right\},$$

being $\mathbf{P}_d$ the space of polynomials of degree not greater than d. The problem of $\epsilon-$interpolation onto S_h^d is defined as follows: given $f : \mathbf{R} \to \mathbf{R}$, 1-periodic, find $f_h \in S_h^d$ such that

$$f_h(x_i + \epsilon h) = f(x_i + \epsilon h), \quad i = 0, \ldots, N-1.$$

In other words, we consider interpolation with nodes $\{x_i + \epsilon h\}_{i=0}^{N-1}$. The parameter ϵ is taken in $[0,1)$.

It is well known (although it will be proven later in this work) that for d even the problem is well posed for $\epsilon \neq 0$ and for d odd if $\epsilon \neq 1/2$. For a discussion of the 'singular cases' see [9]. In the uniquely solvable cases, we define $Q_{h,\epsilon}^d : \mathcal{C} \to S_h^d$ by

$$Q_{h,\epsilon}^d f(x_i + \epsilon h) = f(x_i + \epsilon h), \qquad \forall i \in \mathbf{Z}.$$

The cases (d even/$\epsilon = 1/2$) and (d odd/$\epsilon = 0$) will be referred to as classical interpolation.

In this work we will show the existence of a sequence of polynomials $\{P_{n,\epsilon}^d\}$ such that for u smooth enough

$$Q_{h,\epsilon}^d u - u = \sum_{n=d+1}^{M} h^n P_{n,\epsilon}^d\left(\frac{\cdot - x_i}{h}\right) u^{(n)} + \mathcal{O}(h^{M+1}), \tag{1}$$

uniformly in all the intervals $[x_i, x_{i+1}]$. Moreover, we will show the relationship between the family $P_{n,\epsilon}^d$ and the set of Bernoulli polynomials, plus some symmetries in the classical cases. The proof of this result will be done with Fourier analysis techniques.

2.1 Some functional spaces

In order to deal with the Fourier series without many convergence problems we introduce some notations. Given $u \in L^1(0,1)$, we denote

$$\widehat{u}(k) := \int_0^1 u(x)\phi_{-k}(x)\,dx, \quad k \in \mathbf{Z},$$

being $\phi_k(x) := \exp(2k\pi\imath x)$. For $m \in \mathbf{N} \cup \{0\}$ we define the set

$$W_m := \left\{ u \in L^1(0,1) \,\middle|\, \sum_{k \neq 0} |k|^m |\widehat{u}(k)| \leq \infty \right\}.$$

It follows readily that W_m is a Banach space when endowed with the norm

$$\|u\|_m := |\widehat{u}(0)| + \sum_{k \neq 0} |k|^m |\widehat{u}(k)|.$$

We remark that W_0 is the Wiener algebra. If we denote $\|g\|_\infty := \max_{0 \leq t \leq 1} |g(t)|$, then W_m is continuously embedded in the space $\mathcal{C}^m$ where the usual norm

$$\max_{0 \leq j \leq m} \|u^{(j)}\|_\infty$$

is considered. Moreover, the Fourier series of $u \in W_m$ converges in the norm of $\mathcal{C}^m$.

2.2 Periodic interpolation via Fourier analysis

Periodic splines can be identified by recursion properties of their Fourier coe□cients. In fact, given $u \in L^1(0,1)$, it is easily proven by induction (see also [2]) that $u \in S_h^d$ if and only if

$$(k + pN)^{d+1}\widehat{u}(k + pN) = k^{d+1}\widehat{u}(k), \quad \forall k, p \in \mathbf{Z}. \tag{2}$$

Therefore, periodic splines are uniquely determined by N consecutive Fourier coe□cients, which are themselves free to be chosen. Let then

$$\Lambda_N := \left\{ \mu \in \mathbf{Z} \,\middle|\, -\frac{N}{2} < \mu \leq \frac{N}{2} \right\}$$

be a set of representatives of integers modulo N. The operator

$$\begin{array}{ccc} S_h^d & \longrightarrow & \mathbf{C}^N \\ u_h & \longmapsto & (\widehat{u}_h(\mu))_{\mu \in \Lambda_N} \end{array} \tag{3}$$

is hence an isomorphism and induces a Lagrange basis of S_h^d, $\{\varphi_\mu^N\}_{\mu \in \Lambda_N}$ given by

$$\widehat{\varphi}_\mu^N(m) = \begin{cases} 1, & \mu = m, \\ 0, & \mu \neq m, \end{cases} \qquad \mu, m \in \Lambda_N.$$

Therefore, by (2)

$$\begin{aligned}\varphi_\mu^N &= \sum_{k\in\mathbf{Z}} \widehat{\varphi}_\mu^N(k)\phi_k = \sum_{k\in\mathbf{Z}} \widehat{\varphi}_\mu^N(\mu+kN)\phi_{\mu+kN} \\ &= \left(1+\sum_{k\neq 0}\left(\frac{\mu}{\mu+kN}\right)^{d+1}\phi_k(N\,\cdot\,)\right)\phi_\mu\end{aligned}$$

with absolute and uniform convergence of the series (since $d\geq 1$). Let

$$\Delta_d(x,y) := y^{d+1}\sum_{k\neq 0}\frac{1}{(k+y)^{d+1}}\phi_k(x),$$

where the series is absolutely convergent, $y-$uniformly in compact sets of $(-1,1)$. We have proven that

$$\varphi_\mu^N = \left(1+\Delta_d\left(Nx,\frac{\mu}{N}\right)\right)\phi_\mu, \qquad \forall \mu\in\Lambda_N. \tag{4}$$

Lemma 2.1 *For $u\in W_0$, the $\epsilon-$interpolation problem onto S_h^d is equivalent to*

$$\left|\begin{array}{l} u_h = \displaystyle\sum_{\mu\in\Lambda_N}\widehat{u}_h(\mu)\varphi_\mu^N, \\ \left(1+\Delta_d\left(\epsilon,\dfrac{\mu}{N}\right)\right)\widehat{u}_h(\mu) = \displaystyle\sum_{k\in\mathbf{Z}}\widehat{u}(\mu+kN)\phi_k(\epsilon), \quad \forall\mu\in\Lambda_N.\end{array}\right. \tag{5}$$

Proof. Since φ_μ^N is a Lagrange basis for the operator (3), then for all $u_h\in S_h^d$ we can write

$$u_h = \sum_{\mu\in\Lambda_N}\widehat{u}_h(\mu)\varphi_\mu^N.$$

By (4),

$$u_h(x_i+\epsilon h) = \sum_{\mu\in\Lambda_N}\widehat{u}_h(\mu)\left(1+\Delta_d\left(\epsilon,\frac{\mu}{N}\right)\right)\phi_\mu(x_i+\epsilon h).$$

On the other hand,

$$u(x_i+\epsilon h) = \sum_{j\in\mathbf{Z}}\widehat{u}(j)\phi_j(x_i+\epsilon h) = \sum_{\mu\in\Lambda_N}\left\{\sum_{k\in\mathbf{Z}}\widehat{u}(\mu+kN)\phi_k(\epsilon)\right\}\phi_\mu(x_i+\epsilon h).$$

The matrix $(\phi_\mu(x_i+\epsilon h))_{i,\mu\in\Lambda_N}$ can be easily decomposed as the product of the Vandermonde matrix $(\phi_\mu(x_i))$ by the diagonal matrix $\mathrm{diag}\,(\phi_\mu(\epsilon h))$. Then the system of equations $u_h(x_i+\epsilon h)=u(x_i+\epsilon h)$ for all $i\in\mathbf{Z}$ is equivalent to (5). □

By Theorem 2 of [6] we have that $|1+\Delta_d(x,y)|=0$ if and only if

$$\begin{aligned}(x,y) &= (0,1/2) \text{ and } d \text{ even, or}\\ (x,y) &= (1/2,1/2) \text{ and } d \text{ odd.}\end{aligned}$$

Hence, $\epsilon-$interpolation with $\epsilon\neq 0$ (resp. $\epsilon\neq 1/2$) is well defined for d odd (resp. d even).

3 The local expansion

In this section we will prove (1). An important role will be played by the Bernoulli polynomials $\{B_n\}_n$. For an introduction to this family of polynomials, see [1] and [17]. We begin with some technical lemmas.

Lemma 3.1 *Let $d \geq 1$. Then, uniformly for all $(x, y) \in \mathbf{R} \times [-1/2, 1/2]$,*

$$\Delta_d(x, y) = -\sum_{n=d+1}^{M} \frac{1}{n!} \binom{-(d+1)}{n-(d+1)} y^n (2\pi\imath)^n \underline{B}_n(x) + \mathcal{O}(y^{M+1}).$$

Proof. Since Δ_d is 1−periodic in its first variable we can restrict our attention to $(x, y) \in [0, 1] \times [-1/2, 1/2]$. From the definition of Δ_d, we obtain

$$\begin{aligned} \Delta_d(x, y) &= y^{d+1} \sum_{k \neq 0} \frac{1}{k^{d+1}} \left(1 + \frac{y}{k}\right)^{-(d+1)} \phi_k(x) \\ &= y^{d+1} \sum_{k \neq 0} \frac{1}{k^{d+1}} \left\{ \sum_{m=0}^{M} \binom{-(d+1)}{m} \left(\frac{y}{k}\right)^m \right\} \phi_k(x) + \mathcal{O}(y^{M+1}) \qquad (6) \end{aligned}$$

uniform and absolutely for $(x, y) \in [0, 1] \times [-1/2, 1/2]$. Hence, we can reorder the previous series and (6) implies that

$$\Delta_d(x, y) = \sum_{n=d+1}^{M} \binom{-(d+1)}{n-(d+1)} y^n \left\{ \sum_{k \neq 0} \frac{1}{k^n} \phi_k(x) \right\} + \mathcal{O}(y^{M+1}).$$

Finally, by the Fourier expansion of the Bernoulli polynomials (see [1]),

$$B_n = -\frac{n!}{(2\pi\imath)^n} \sum_{k \neq 0} \frac{1}{k^n} \phi_k,$$

the result follows readily. □

Let us consider the operator

$$D_N u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \varphi_\mu^N.$$

Obviously, $u \in S_h^d$ if and only if $D_N u = u$. A consequence of the previous lemma is the following asymptotic expansion of $D_N u$ for su□ciently smooth u.

Corollary 3.2 *Let $u \in W_{M+1}$. Then, uniformly in $\mathbf{R}$, it holds that*

$$D_N u = u - \sum_{n=d+1}^{M} \frac{1}{n!} \binom{-(d+1)}{n-(d+1)} h^n \underline{B}_n(N\,\cdot) u^{(n)} + \mathcal{O}(h^{M+1}) \|u\|_{M+1}.$$

Proof. Consider first the operators of truncation for the Fourier series

$$S_N u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \phi_\mu.$$

It is then straightforward that

$$\|S_N u - u\|_\infty \le \left(\frac{2}{N}\right)^m \|u\|_m, \qquad \forall u \in W_m. \tag{7}$$

By (4), we have

$$D_N u(x) = S_N u(x) + \sum_{\mu\in\Lambda_N} \widehat{u}(\mu)\Delta_{d+1}\left(Nx, \frac{\mu}{N}\right)\phi_\mu(x).$$

Applying Lemma 3.1 to the second term, we obtain

$$\begin{aligned} D_N u(x) &= S_N u(x) - \sum_{n=d+1}^{M} \frac{h^n}{n!}\binom{-(d+1)}{n-(d+1)}\underline{B}_n(Nx)\Big\{\sum_{\mu\in\Lambda_N}(2\pi\imath\mu)^n \widehat{u}(\mu)\phi_\mu(x)\Big\} \\ &\quad + \mathcal{O}(h^{M+1})\|u\|_{M+1} \\ &= S_N u(x) - \sum_{n=d+1}^{M} \frac{h^n}{n!}\binom{-(d+1)}{n-(d+1)}\underline{B}_n(Nx) S_N u^{(n)}(x) + \mathcal{O}(h^{M+1})\|u\|_{M+1}. \end{aligned}$$

The statement then follows because of (7). □

Lemma 3.3 *There exists a sequence of polynomials* $\{I_n^d\}$ *such that if* $\epsilon \in [0,1)$ *satisfies* $|1+\Delta_d(\epsilon,y)| \ne 0$ *for all* $y \in [-1/2, 1/2]$*, the expansion*

$$\frac{1}{1+\Delta_d(\epsilon,y)} = 1 + \sum_{n=d+1}^{M}(2\pi\imath)^n y^n I_n^d(\epsilon) + \mathcal{O}(h^{M+1})$$

holds uniformly for all $y \in [-1/2, 1/2]$.

Proof. Firstly, let us consider the identity

$$\frac{1}{1+\Delta_d(\epsilon,y)} = 1 + \sum_{k=1}^{m}(\Delta_d(\epsilon,y))^k + (-1)^m \frac{(\Delta_d(\epsilon,y))^{m+1}}{1+\Delta_d(\epsilon,y)}.$$

Since $|1+\Delta_d(\epsilon,y)| > 0$ and $|\Delta_d(\epsilon,y)| \le C|y|^{d+1}$, the bound for the remainder

$$|(\Delta_d(\epsilon,y))^{m+1}(1+\Delta_d(\epsilon,y))^{-1}| \le C|y|^{m(d+1)},$$

follows readily. On the other hand, we can apply Lemma 3.1 to each term of the sum. Reordering the sums we obtain the result. □

Now we are able to state and prove the main result of this section.

Proposition 3.4 *There exists a sequence of polynomials* $\{P_{k,\epsilon}^d\}_k$*, depending only on* d *and* ϵ *such that*

$$Q_{h,\epsilon}^d u(x) - u(x)\Big|_{[x_i,x_{i+1}]} = \sum_{k=d+1}^{M} h^k P_{k,\epsilon}^d\left(\frac{x-x_i}{h}\right)u^{(k)}(x) + \mathcal{O}(h^{M+1})\|u\|_{M+1}, \tag{8}$$

uniformly in $\mathbf{R}$*. Moreover, the degree of* $P^d_{k,\epsilon}$ *is* k *and*

$$P^d_{k,\epsilon}(\epsilon) = 0, \quad \forall k, \tag{9}$$
$$P^d_{k,\epsilon}(x) = -\frac{1}{k!}\binom{-(d+1)}{k-(d+1)}(B_k(x)-B_k(\epsilon)), \quad d+1 \le k \le 2d+1. \tag{10}$$

Finally, in the case of 'classical' periodic interpolation ($\epsilon = 0$ *for* d *odd,* $\epsilon = 1/2$ *for* d *even), we have*

$$P^d_{k,\epsilon}(1-x) = (-1)^k P^d_{k,\epsilon}(x). \tag{11}$$

Proof. By Lemma 2.1, we have for $u \in W_{M+1}$

$$Q^d_{h,\epsilon}u = \sum_{\mu\in\Lambda_N}\left(1+\Delta_d\left(\epsilon,\frac{\mu}{N}\right)\right)^{-1}\Big\{\sum_{k\in\mathbf{Z}}\widehat{u}(\mu+kN)\phi_k(\epsilon)\Big\}\varphi^N_\mu. \tag{12}$$

Since for $\mu \in \Lambda_N$,

$$\Big|\sum_{k\neq 0}\widehat{u}(\mu+kN)\phi_k(\epsilon)\Big| \le Ch^{M+1}\|u\|_{M+1},$$

we can ignore the coefficients $\widehat{u}(m)$ for $m \notin \Lambda_N$ in (12). Also, applying Lemma 3.3 and Corollary 3.2 in the same expression, we obtain

$$\begin{aligned} Q^d_{h,\epsilon}u &= \sum_{\mu\in\Lambda_N}\Big\{1+\sum_{n=d+1}^{M}\left(\frac{\mu}{N}\right)^n(2\pi\imath)^n I^d_n(\epsilon)\Big\}\widehat{u}(\mu)\varphi_\mu + \mathcal{O}(h^{M+1})\|u\|_{M+1} \\ &= D_N u + \sum_{n=d+1}^{M} h^n I^d_n(\epsilon) D_N u^{(n)} + \mathcal{O}(h^{M+1})\|u\|_{M+1} \\ &= u + \sum_{k=d+1}^{M} h^k \underline{P}^d_{k,\epsilon}(N\,\cdot)u^{(k)} + \mathcal{O}(h^{M+1})\|u\|_{M+1}, \end{aligned}$$

where

$$P^d_{n,\epsilon} := -\binom{-(d+1)}{n-(d+1)}\frac{1}{n!}B_n - \sum_{\substack{j+k=n\\ j,k\ge d+1}} I^d_j(\epsilon)\frac{1}{k!}\binom{-(d+1)}{k-(d+1)}B_k + I^d_n(\epsilon).$$

Thus, the degree of $P^d_{n,\epsilon}$ is n. Notice that if $x \in [x_i, x_{i+1}]$, then

$$\underline{P}^d_{k,\epsilon}(Nx) = P^d_{k,\epsilon}\left(\frac{x-x_i}{h}\right)$$

and hence (8) is proven. In addition to this, since for $d+1 \le n \le 2d+1$

$$I^d_n(\epsilon) = \frac{1}{n!}\binom{d+1}{n-(d+1)}B_n(\epsilon)$$

(see Lemma 3.1 and the proof of Lemma 3.3), then (10) follows readily. Furthermore, (9) is a direct consequence of the identity

$$0 = Q^d_{h,\epsilon}u(x_i+\epsilon h) - u(x_i+\epsilon h) = \sum_{j=d+1}^{M} h^k P^d_{k,\epsilon}(\epsilon)u^{(k)}(x_i+\epsilon h) + \mathcal{O}(h^{d+1})\|u\|_{M+1}$$

valid for all $u \in W_{M+1}$.

Suppose now that we are in the classical case, where for all i, $1 - x_i - \epsilon h$ is an interpolation node. We take $u \in W_{M+1}$ and define $\widetilde{u} \in W_{M+1}$ by $\widetilde{u} := u(1 - \cdot)$. The equality

$$Q^d_{h,\epsilon}\widetilde{u}(x_i + \epsilon h) = \widetilde{u}(x_i + \epsilon h) = u(1 - x_i - \epsilon h) = Q^d_{h,\epsilon}u(1 - x_i - \epsilon h), \qquad \forall i \in \mathbf{Z}$$

implies that $Q^d_{h,\epsilon}u(1 - x) = Q^d_{h,\epsilon}\widetilde{u}(x)$, Finally, by the asymptotic expansion we obtain

$$\begin{aligned} 0 &= Q^d_{h,\epsilon}u(1 - x) - Q^d_{h,\epsilon}\widetilde{u}(x) \\ &= \sum_{k=d+1}^{M} h^k\Big(\underline{P}^d_{k,\epsilon}(-Nx) + (-1)^k \underline{P}^d_{k,\epsilon}(Nx)\Big)u^{(k)}(x) + \mathcal{O}(h^{M+1})\|u\|_{M+1}, \end{aligned}$$

and therefore, the symmetries (11) follow readily. □

4 Applications to integral equations of the second kind

We will apply the earlier expansion to study the $\epsilon-$collocation method applied to integral equations of the second kind with smooth periodic kernels. We begin by introducing the problem. Let K be an operator of the form

$$Ku := \int_0^1 K(\cdot, x)u(x)\,dx,$$

where K is $\mathcal{C}^\infty$ and $1-$periodic in both variables. Given a $1-$periodic function f we consider the integral equation of the second kind

$$u + Ku = f. \tag{13}$$

If I is the identity operator, we assume that $(I + K)u = 0$ implies that $u = 0$. Because of the Fredholm alternative this proves that $I + K : W_n \to W_n$ has a bounded inverse for all n.

Given $\epsilon \in [0, 1)$, the $\epsilon-$collocation method for (13) is

$$\left| \begin{array}{l} u_h \in S^d_h \\ (I + K)u_h(x_i + \epsilon h) = f(x_i + \epsilon h), \quad i = 0, \ldots, N - 1. \end{array} \right. \tag{14}$$

We denote

$$\zeta(d) := \begin{cases} 0, & d \text{ odd} \\ \frac{1}{2}, & d \text{ even} \end{cases}$$

Then, if $\epsilon = \zeta(d)$, (14) is the collocation method studied in [3] and [12] in the wider context of pseudodifferential equations on smooth closed curves. The general case has been studied in [14]. In this paper, it is proven that $\epsilon-$collocation is stable and convergent in suitable Sobolev norms if and only if $\epsilon \neq 0$ and d odd or $\epsilon \neq 1/2$ and d even.

Notice that $u_h \in S_h^d$ is the solution to (14) if and only if

$$Q_{h,\epsilon}^d(I + K)u_h = Q_{h,\epsilon}^d(I + K)u. \tag{15}$$

Then (15) induces a map $C_{h,\epsilon}^d : W_0 \to S_h^d$, by $C_{h,\epsilon}^d u := u_h$.

For convenience we work with the periodic Sobolev spaces, where the analysis of these methods has been carried out. Let $\mathbf{T} := \mathbf{C}\langle \phi_k \mid k \in \mathbf{Z}\rangle$ be the space of trigonometric polynomials. For $s \in \mathbf{R}$ we define the norm in $\mathbf{T}$

$$\|u\|_{s,2} := \Big(|\widehat{u}(0)|^2 + \sum_{k\neq 0} |k|^{2s}|\widehat{u}(k)|^2\Big)^{1/2}.$$

Let then H^s be the closure of $\mathbf{T}$ with this norm. Obviously, H^0 can be identified with $L^2(0,1)$, their norms being equal, and for $s > 0$, H^s can be identified with the space of functions $u \in L^2(0,1)$ such that $\|u\|_{s,2} < \infty$. Moreover, for all $s \in \mathbf{R}$, H^s and H^{-s} form a dual pair via the extension of the inner product of H^0

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}dx$$

and

$$\|u\|_{s,2} = \sup_{0\neq v\in H^{-s}} \frac{|\langle u, v\rangle|}{\|v\|_{-s,2}}.$$

Finally, we have

$$H^{n+1/2+\delta} \subset W_n \subset H^n, \qquad \forall\delta > 0, \quad n \in \mathbf{N}\cup\{0\}, \tag{16}$$

the injections being bounded. Moreover $S_h^d \subset H^t$ for $t < d + 1/2$.

We take $Q_{h,\epsilon}^d : W_0 \to S_h^d$. Then it is proven in [14] (although it can be easily deduced from Lemma 2.1 and some properties of splines) that for h small enough and $s \in (1/2, d + 1/2)$, there exists C, depending only on s and ϵ, such that for all $u \in H^s$, $\|Q_{h,\epsilon}^d u\|_{s,2} \leq C\|u\|_{s,2}$. From standard arguments about compact perturbations of projection methods (cf. [8], [9]), we can extend this stability property to $C_{h,\epsilon}^d$ and obtain

$$\|C_{h,\epsilon}^d u\|_{s,2} \leq C\|u\|_{s,2}, \qquad \forall u \in H^s, \tag{17}$$

for $s \in (1/2, d + 1/2)$, with C depending on s and ϵ.

The asymptotic analysis to be developed in the next pages follows a quite straightforward plan: obtain consitency expansions, write stability in the form of a suitable inf–sup condition, deduce convergence estimates and finally new asymptotic properties as a consequence.

4.1 Asymptotic analysis of the $\epsilon-$collocation

The main result of this section is the existence of a uniform asymptotic expansion of the difference between the interpolate of the solution $Q_{h,\delta}^d u$ and the numerical solution $C_{h,\epsilon}^d u$. A priori we take $\epsilon \neq \delta$.

If $\{P_{k,\delta}^d\}$ is the sequence of polynomials of Proposition 3.4, we define

$$\alpha_{k,\delta}^d := \int_0^1 P_{k,\delta}^d(x)\, dx.$$

Lemma 4.1 *Let $u \in W_{M+1}$. Then, uniformly in $\mathbf{R}$, we have*

$$K(Q^d_{h,\delta}u - u) = \sum_{k=d+1}^{M} \alpha^d_{k,\delta} h^k K u^{(k)} + \mathcal{O}(h^{M+1})\|u\|_{M+1}. \tag{18}$$

Proof. Let $f \in L^1(0,1)$ be 1−periodically extended to $\mathbf{R}$. Then we can write

$$\int_0^1 K(\cdot,x) f(Nx) u(x)\, dx \;=\; \sum_{j=1}^{N} \int^{x_{j+1}} \quad -$$

Proposition 4.3 *Let $\epsilon \in [0,1)$ be such that the $\epsilon-$collocation method is stable. Then for $s \in (0,d)$ there exists $\beta = \beta(s,\epsilon)$, such that for h small enough*

$$\inf_{u_h \in S_h^d} \sup_{0 \neq t_h \in T_h} \frac{|\langle (I+K)u_h, t_h\rangle|}{\|u_h\|_{\frac{1}{2}+s,2}\|t_h\|_{-\frac{1}{2}-s,2}} > \beta.$$

Proof. We follow closely the proof of Proposition 8 in [5]. For $s \in (1/2, d+1/2)$, we consider the H^0-adjoint operator of $Q^d_{h,\epsilon} : H^s \to H^s$, i.e., $Q^{d,*}_{h,\epsilon} : H^{-s} \to H^{-s}$ satisfying

$$\langle Q^d_{h,\epsilon} u, t\rangle = \langle u, Q^{d,*}_{h,\epsilon} t\rangle, \qquad \forall u \in H^s, \forall t \in H^{-s}.$$

It is then easy to prove that $Q^{d,*}_{h,\epsilon}(H^{-s}) = T_h$.

Given $v_h \in S^d_h$, we take $w_h := Q^d_{h,\epsilon}(I+K)v_h$. Then

$$Q^d_{h,\epsilon}(I+K)v_h = w_h = Q^d_{h,\epsilon}(I+K)(I+K)^{-1}w_h$$

and therefore, $v_h = C^d_{h,\epsilon}(I+K)^{-1}w_h$. Thus, $\|v_h\|_{\frac{1}{2}+s,2} \leq C\|w_h\|_{\frac{1}{2}+s,2}$ for all $s \in (0,d)$. On the other hand,

$$\begin{aligned}
\|w_h\|_{\frac{1}{2}+s,2} &= \sup_{0\neq t \in H^{-1/2-s}} \frac{|\langle w_h, t\rangle|}{\|t\|_{-\frac{1}{2}-s,2}} = \sup_{\substack{t\in H^{-1/2-s}\\ Q^{d,*}_{h,} t \neq 0}} \frac{|\langle (I+K)v_h, Q^{d,*}_{h,\epsilon} t\rangle|}{\|Q^{d,*}_{h,\epsilon} t\|_{-\frac{1}{2}-s,2}} \frac{\|Q^{d,*}_{h,\epsilon} t\|_{-\frac{1}{2}-s,2}}{\|t\|_{-\frac{1}{2}-s,2}} \\
&\leq \|Q^{d,*}_{h,\epsilon}\| \sup_{0\neq t_h \in T_h} \frac{|\langle (I+K)v_h, t_h\rangle|}{\|t_h\|_{-\frac{1}{2}-s,2}}.
\end{aligned}$$

Now the result follows readily by the uniform boundedness of $Q^d_{h,\epsilon}$. □

We are now ready to formulate and prove the asymptotic expansion.

Proposition 4.4 *Let $\epsilon, \delta \in [0,1)$ such that the operator $C^d_{h,\epsilon}, Q^d_{h,\delta} : W_0 \to S^d_h$ are well define. Then for $u \in W_{M+1}$ the following expansion holds*

$$\Big\| C^d_{h,\epsilon} u - Q^d_{h,\delta} u + \sum_{k=d+1}^{M} h^k \Big\{ \alpha^d_{k,\delta} C^d_{h,\epsilon} v_k + P^d_{k,\delta}(\epsilon) C^d_{h,\epsilon} w_k \Big\} \Big\|_0 \leq C h^{M+t} \|u\|_{M+1},$$

for all $t < 1/2$, with $w_k := (I+K)^{-1}u^{(k)}$ and $v_k := K w_k$.

Proof. Let $t_h := \sum_{i=1}^N t_i \delta_{i,\epsilon}, \in T_h$. Then, denoting v_k and w_k as in the statement of the proposition, by (21) it follows that

$$\Big|\Big\langle (I+K)\Big(u - Q^d_{h,\delta} u + \sum_{k=d+1}^{M} \Big\{ \alpha_{k,\delta} h^k v_k + P_{k,\delta}(\epsilon) h^k w_k \Big\}\Big), t_h \Big\rangle\Big| \leq C h^{M+1} \|u\|_{M+1} \sum_{i=1}^N |t_i|,$$

since $K(I+K)^{-1} = (I+K)^{-1}K$. For $s \in (0, 1/2]$ there exists $C > 0$ independent of h such that

$$h^{1/2+s} \sum_{i=1}^N |t_i| \leq C \|t_h\|_{-\frac{1}{2}-s,2}, \qquad \forall t_h = \sum_{i=1}^N t_i \delta_{i,\epsilon} \in T_h. \tag{22}$$

This bound is proven for $s = 1/2$ in [5] Lemma 9. For the other cases, the result follows by the same techniques.

Then the first inequality of this proof, Proposition 4.3, (22) and the continuous embedding of $H^{1/2+s}$ in W_0 prove the result. □

For simplicity we will require a little more regularity on the function u, namely $u \in W_{M+2}$. Then, taking an additional term in Proposition 4.4

$$\left\| C^d_{h,\epsilon} u - Q^d_{h,\delta} u + \sum_{k=d+1}^{M} \alpha^d_{k,\delta} h^k C^d_{h,\epsilon} v_k + \sum_{k=d+1}^{M} P^d_{k,\delta}(\epsilon) h^k C^d_{h,\epsilon} w_k \right\|_0 \le C h^{M+1} \|u\|_{M+2},$$

since

$$\|C^d_{h,\epsilon} v_{M+1}\|_0 + \|C^d_{h,\epsilon} w_{M+1}\|_0 \le C(\|w_{M+1}\|_1 + \|v_{M+1}\|_1) \le C' \|u\|_{M+2}. \tag{23}$$

The first inequality in (23) follows from (16) and (17), whereas the second one is a consequence of the definition of v_{M+1} and w_{M+1} and the fact that differentiation is bounded from W_k to W_{k-1} for all k. We emphasize that all the following results can be adapted to the case $u \in W_{M+1}$.

Comparison of $C^d_{h,\epsilon} u$ with $Q^d_{h,\epsilon} u$ (i.e., $\epsilon = \delta$) yields

$$\left\| C^d_{h,\epsilon} u - Q^d_{h,\epsilon} u + \sum_{k=d+1}^{M} \alpha^d_{k,\delta} h^k C^d_{h,\epsilon} (I + K)^{-1} K u^{(k)} \right\|_0 \le C h^{M+1} \|u\|_{M+2}.$$

Some particular choices of ϵ, δ give special properties to the expansion of Proposition 4.4. In view of Remark 4.2, the roots of Bernoulli polynomials will be of importance. It can be easily proven by induction that for $n > 0$ the only roots in $[0, 1]$ of B_{2n+1} are $0, 1$ and $1/2$. On the other hand, the polynomials B_{2n} have only two roots in the same interval, which will be denoted by $\xi_1(2n) < \xi_2(2n)$, and are symmetrically disposed with respect to $1/2$.

We finish this section with some superconvergence properties of the ϵ-collocation method.

Corollary 4.5 *We have*

$$\left\| C^d_{h,\epsilon} u - Q^d_{h,\delta} u \right\|_0 \;\le\; C h^{d+1} \|u\|_{d+2}.$$

Moreover, for d even

$$\left\| C_{h,1/2} u - Q_{h,1/2} u \right\|_0 \;\le\; C h^{d+2} \|u\|_{d+3},$$

whereas for d odd,

$$\left\| C^d_{h,\xi_i(d+1)} u - Q^d_{h,\xi_j(d+1)} u \right\|_0 \;\le\; C h^{d+2} \|u\|_{d+3}, \quad i, j = 1, 2.$$

Proof. The first estimate is a direct consequence of Proposition 4.4. The second and the third ones follow from the fact that the expansion begins with h^{d+2}. □

4.2 Further expansions

This section deals with a final expansion of error when we apply a smooth operator to u_h. This can have important applications, as for example, in the Boundary elements method, when we calculate the potential in points of the domain, via the integral representation. In this case, if the curve is regular, the kernel of the integral representation is smooth.

Let Ω be an open set in the plane and $T : \Omega \times \mathbf{R} \to \mathbf{C}$, smooth and $1-$periodic in the second variable. Let us consider the operator $T : L^1(0,1) \to \mathcal{C}^\infty(\Omega)$,

$$Tu := \int_0^1 T(\cdot, x)u(x)dx.$$

In $\mathcal{C}^\infty(\Omega)$, we consider the usual topology of uniform convergence of all derivatives on compact sets (cf. [10]). Let

$$p(d) := \begin{cases} d+1, & \text{if } d \text{ odd,} \\ d+2, & \text{if } d \text{ even.} \end{cases}$$

Proposition 4.6 *For $u \in W_{M+2}$, there exists a sequence of functions $\{u_k\}_k$ independent of h such that*

$$T(C^d_{h,\epsilon}u - u) = \sum_{k=d+1}^{M} h^k T u_k + \mathcal{O}(h^{M+1})\|u\|_{M+2}. \tag{24}$$

Moveover, if $u \in W_{2M+3}$ and we choose $\epsilon = \zeta(d)$ (the classical collocation method)

$$T(C^d_{h,\zeta(d)}u - u) = \sum_{k=p(d)/2}^{M} h^{2k} T u_k + \mathcal{O}(h^{2M+2})\|u\|_{2M+3}. \tag{25}$$

Finally, for d odd and $\epsilon = \xi_j(d+1)$, the expansion (24) takes the form

$$T(C^d_{h,\xi_j(d+1)}u - u) = \sum_{k=d+2}^{M} h^k T u_k + \mathcal{O}(h^{M+1})\|u\|_{M+2}, \quad j = 1, 2.$$

All the equalities hold in $\mathcal{C}^\infty(\Omega)$.

Proof. With the same techniques as those used in Lemma 4.1 it can be proved that for $u \in W_{M+1}$

$$T(Q^d_{h,\delta}u - u) = \sum_{k=d+1}^{M} \alpha^d_{k,\delta} h^k T u^{(k)} + \mathcal{O}(h^{M+1})\|u\|_{M+1}.$$

Then, from Proposition 4.4 (we recall that $v_k := (I + K)^{-1} K u^{(k)}$), it follows that

$$\begin{aligned} T(C^d_{h,\epsilon}u - u) &= T(C^d_{h,\epsilon}u - Q^d_{h,\epsilon}u) + T(Q^d_{h,\epsilon}u - u) \\ &= -\sum_{k=d+1}^{M} \alpha^d_{k,\epsilon} h^k T C^d_{h,\epsilon} v_k + \sum_{k=d+1}^{M} \alpha^d_{k,\epsilon} h^k T u^{(k)} + \mathcal{O}(h^{M+1})\|u\|_{M+2} \\ &= \sum_{k=d+1}^{M} \alpha^d_{k,\epsilon} h^k T(v_k - C^d_{h,\epsilon} v_k) + \sum_{k=d+1}^{M} \alpha^d_{k,\epsilon} h^k T(u^{(k)} - v_k) \\ &\quad + \mathcal{O}(h^{M+1})\|u\|_{M+2}. \end{aligned}$$

This equality and the identity $I - (I+K)^{-1}K = (I+K)^{-1}$ yield

$$T(C^d_{h,\epsilon}u - u) = \sum_{k=d+1}^{M} \alpha^d_{k,\epsilon} h^k \left\{ T(v_k - C^d_{h,\epsilon} v_k) + T(I+K)^{-1} u^{(k)} \right\} + \mathcal{O}(h^{M+1}) \|u\|_{M+2}.$$

Thus, (24) follows readily by induction.

Notice that for $\epsilon = \zeta(d)$, $\alpha_{2k+1,\zeta(d)}=0$ and the previous expansion has only even powers of h. Finally, by definition of $\xi_j(d+1)$ and Remark 4.2, we have $\alpha_{d+1,\xi_j(d+1)} = 0$ for d odd and $j = 1, 2$, and therefore, the expansion begins in h^{d+2}. □

Remark 4.7 *Proposition 4.6 proves that for d odd, there are two points, the roots of B_{d+1} in $(0,1)$, that provide an additional order of convergence.*

Remark 4.8 *The Sloan iteration (cf. [4] [15]) of the numerical solution to $u + Ku = f$ is defined by*

$$u_h^* := f - Ku_h.$$

Since $u - u_h^* = K(u_h - u)$ we can follow step by step the proof of Proposition 4.6 to obtain the same kind of expansions in any of the norms of W_n, since $K : W_0 \to W_n$ is always bounded. This is a traditional way of both smoothing the solution (notice that u_h^* is no longer a spline, but a smooth function) and obtaining optimal convergence properties.

Remark 4.9 *From Theorem 3.3' in [11], a result similar to (25) can be proven, being T a linear functional of the form $\langle a, \cdot \rangle$ with a periodic and $\mathcal{C}^\infty$. Because of specialising to a particular case, our results are stronger and the proofs simpler.*

5 Some numerical tests

In this section we test some of the results obtained for integral equations with a simple case, using polygonals as trial functions in the $\epsilon-$collocation method. The following example is taken from [8]. Let Ω be a bounded domain in $\mathbf{R}^2$ with smooth boundary $\Gamma := \partial\Omega$. We consider the Dirichlet problem for Laplace's equation

$$\left| \begin{array}{l} \Delta\omega = 0, \quad \text{in } \Omega, \\ \omega|_\Gamma = f. \end{array} \right.$$

The potential ω can be calculated by means of the double layer potential

$$\omega(\boldsymbol{x}) = -\frac{1}{2\pi} \int_\Gamma u(\boldsymbol{y}) \frac{\partial}{\partial n_{\boldsymbol{y}}} \log|\boldsymbol{x} - \boldsymbol{y}|\, d\sigma_{\boldsymbol{y}} =: Du(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

where u is solution of the integral equation

$$u(\boldsymbol{x}) + \frac{1}{\pi} \int_\Gamma u(\boldsymbol{y}) \frac{\partial}{\partial n_{\boldsymbol{y}}} \log|\boldsymbol{x} - \boldsymbol{y}|\, d\sigma_{\boldsymbol{y}} = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma. \tag{26}$$

being $g := -2f$. Let $\boldsymbol{x} : [0,1] \to \Gamma$ be a $1-$periodic and regular parameterization of Γ. We identify $f(t) := f(\boldsymbol{x}(t))$, $u(t) := u(\boldsymbol{x}(t))$.

Let Γ be an ellipse with semiaxes $a \geq b > 0$ and $\boldsymbol{x}(t) := (a\cos(2\pi t), b\sin(2\pi t))$. Under these conditions, equation (26) becomes

$$u(t) + ab\int_0^1 \frac{u(s)}{a^2 + b^2 - (a^2 - b^2)\cos(2\pi(s+t))}\, ds = g(t$$

From Proposition 4.6, the asymptotic expansion adopts the form

$$\omega(\boldsymbol{x}) - \omega_h(\boldsymbol{x}) = \sum_{k=1}^{M} h^{2k} u_k(\boldsymbol{x}) + \mathcal{O}(h^{2M+2})\|u\|_{2M+3},$$

uniformly in compact sets of Ω. Since $N_{j+1}/N_j = 3/2$, by extrapolation we obtain a new set of approximate solutions:

$$\begin{cases} \omega_h^0 := \omega_h, \\ \omega_h^j := \dfrac{(9/4)^j \omega_{2h/3}^{j-1} - \omega_h^{j-1}}{(9/4)^j - 1}, \quad j > 0. \end{cases}$$

Table 2 shows the errors at the point $(0,0)$

$$e_h^j := |\omega_h^j(0,0) - \omega(0,0)|,$$

and Table 3, the corresponding estimated convergence rates, with consecutive discretization levels. The expected values in this table are given in the first row

N	e_h^0	e_h^1	e_h^2	e_h^3	e_h^4	e_h^5	e_h^6
64	9.56E-04						
96	4.25E-04	3.00E-07					
144	1.89E-04	5.93E-08	4.79E-11				
216	8.40E-05	1.17E-08	4.21E-12	7.77E-15			
324	3.73E-05	2.31E-09	3.70E-13	3.03E-16	1.92E-19		
486	1.66E-05	4.57E-10	3.25E-14	1.18E-17	3.35E-21	2.09E-23	
729	7.37E-06	9.02E-11	2.86E-15	4.61E-19	5.82E-23	1.54E-25	7.21E-27

Table 2: Errors after extrapolation in the symmetric case

2	4	6	8	10	12
1.9990					
2.0000	4.0016				
2.0000	4.0008	5.9956			
2.0000	4.0003	5.9982	8.0015		
2.0000	4.0001	5.9992	8.0007	9.99849	
2.0000	4.0000	5.9996	8.0003	9.99360	12.1122

Table 3: E.c.r. for the errors in Table 2

Example 3

We apply again extrapolation to calculate the potential in the superconvergent case $\epsilon = \xi_2(2)$ with the same values of N. The asymptotic expansion has now the form

$$\omega - \omega_h = \sum_{k=3}^{M} h^k v_k + \mathcal{O}(h^{M+1}),$$

which means that the extrapolation scheme to be adopted is

$$\begin{cases} \omega_h^0 := \omega_h, \\ \omega_h^j := \dfrac{(3/2)^{j+2}\omega_{2h/3}^{j-1} - \omega_h^{j-1}}{(3/2)^{j+2} - 1}, \quad j > 0. \end{cases}$$

We measure and tabulate errors in the point $(1/4, 1/4)$. Finally Table 5 shows estimated convergence rates.

N	e_h^0	e_h^1	e_h^2	e_h^3	e_h^4	e_h^5	e_h^6
64	4.59E-7						
96	1.09E-7	3.89E-08					
144	2.65E-8	8.14E-09	5.78E-10				
216	6.67E-9	1.67E-09	8.14E-11	6.10E-11			
324	1.74E-9	3.40E-10	1.12E-11	5.57E-12	2.37E-14		
486	4.66E-10	6.83E-11	1.52E-12	4.99E-13	1.02E-15	3.88E-16	
729	1.29E-10	1.37E-11	2.04E-13	4.42E-14	4.61E-17	1.48E-18	3.69E-19

Table 4: Errors after extrapolation in the superconvergent case

3	4	5	6	7	8
3.5538					
3.4833	3.8555				
3.4013	3.9019	4.8339			
3.3181	3.9339	4.8911	5.9022		
3.2420	3.9556	4.9286	5.9533	7.7514	
3.1778	3.9703	4.9531	5.9764	7.6507	8.0584

Table 5: E.c.r. for the errors in Table 4

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