Functional estimation of spatiotemporal heterogeneities

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Abstract: Heterogeneity analysis is one of the most complex concerns in spatiotemporal statistics, due to the different existing sources of variability. In this paper, a functional approximation to the multiscale structure of a certain class of non-homogeneous (non-stationary) spatiotemporal processes, based on its multiple-window atomic decomposition, is studied. The multiscaling exponent, defining the local regularity and correlation of the process, is estimated from such a decomposition.

Keywords: Local means; Multifractional covariance structures; Multiscale processes; Variable fractal local dimension.

1 Introduction

Different levels of heterogeneity are displayed, for example, by environmental and geophysical phenomena, when they are observed at different scales (from the microscopic to the macroscopic scale). Characteristics referred to a specific scale may not appear at other scales. Then, direct extrapolation of available information, parameter values, estimates, etc., from one scale to a larger or smaller scale is not possible, in general. Additional information on the smaller or larger scale considered is usually needed (see Douglas and Barros, 2003; Gaonach, Lovejoy and Schertzer, 2003; Harris et al., 2001; Vörös et al., 2003, among others).

The atomic decomposition allows the representation of discrete (univariate and multivariate) and continuous multiscaling (Ruiz-Medina, Anh and Angulo, 2004a), providing an interpolation between the continuous and discrete wavelet transforms (see, for example, Triebel, 1997; Anh, Angulo and Ruiz-Medina, 2005; Ruiz-Medina, Anh and Angulo, 2004b). In this paper, we derive a multiple window version of such a decomposition to estimate spatiotemporal varying scaling exponents. Specifically, the identification of the underlying non-stationary (in time and space) spatiotemporal process is achieved in terms of a sequence of spatial varying scaling exponents that characterize the evolution of its local regularity and correlation properties. Our aim is to estimate the sequence \( \{H_t(x); x \in D \subseteq \mathbb{R}^n, i = 1, \ldots, k \} \) of spatial varying scaling exponents corresponding to the time instants
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t_1, \ldots, t_k, for a fixed k \in \mathbb{N}_+$. The estimation procedure proposed extends
the multiple-window scalogram analysis, based on the multiple-window
wavelet decomposition, to the atomic decomposition case.

2 Model family

Our study is focused on heterogeneous spatiotemporal processes with spa-
tiotemporal varying local regularity and correlation properties. In particu-
lar, their quadratic variation exponents depend on the location and time.
Examples of such processes can be generated from the theory of pseudodif-
ferential operators of variable order. For instance, we can consider the fam-
ilies of models given by the following integral equations:

\begin{align}
Y_1(t, x) &= \int_0^t \int_{\mathbb{R}^n} \exp \{i \langle x, \lambda \rangle \} p_1(t - s, x, \lambda) \hat{\varepsilon}(s, d\lambda) ds \\
Y_2(t, x) &= \int_{\mathbb{R}^n} \exp \{i \langle x, \lambda \rangle \} p_2(t, x, \lambda) \hat{X}(d\lambda),
\end{align}

where \( p_1 \) and \( p_2 \) respectively represent the symbols of spatiotemporal pseudodif-
ferential operators \( P_1 \) and \( P_2 \), satisfying suitable regularity and moment
conditions, \( \hat{\varepsilon} \) represents white noise, and \( \hat{X} \) represents the spectral process
defining the second-order structure of a homogeneous spatial process \( X \).
The spatial covariance structure of processes \( Y_1 \) and \( Y_2 \) are formally de-
fined from equations (1) and (2) as

\begin{align}
B_{Y_1}(t, x, y) &= \int_0^t \int_{\mathbb{R}^n} \exp \{i \langle x - y, \lambda \rangle \} p_1(t - s, x, \lambda) p_1(t - s, y, \lambda) d\lambda ds \\
B_{Y_2}(t, x, y) &= \int_{\mathbb{R}^n} \exp \{i \langle x - y, \lambda \rangle \} p_2(t, x, \lambda) p_2(t, y, \lambda) f_X(\lambda) d\lambda,
\end{align}

where \( f_X \) denotes the spectral density of the homogeneous spatial process X. Particular choices of \( p_i \), \( i = 1, 2 \), lead to different variable order covariance structures. For \( p_1(t, x) = \exp \{-tp(x, \lambda)\} \), with \( p \) the symbol defining a spatial pseudodifferential operator \( P \) of variable order, we cover the class of stationary in time and heterogeneous in space processes defined as the solution to the multifractional evolution equation

\[ \frac{\partial}{\partial t} Y_1 + PY_1(t, x) = \varepsilon(t, x). \]

In this special case, particular choices of \( p \) are given, for instance, by \( p(x, \lambda) = |\lambda|^{\alpha(x)} \) and \( p(x, \lambda) = (1 + |\lambda|^2)^{-\gamma(x)} \), which respectively generalize the characteristic exponent of Lévy processes, and the Bessel probability distribution. The associated spatial quadratic variation (multiscaling) exponents are then respectively given by \( H_t(x) = \frac{\alpha(x) - n}{2} \) and \( H_t(x) = \)
γ(x) − n, for any t. They also define the local regularity of sample paths in the Gaussian case, when ε is Gaussian white noise. For non-stationarity in time and space, just as an example, we can consider p(t, x) = \exp \{-p(t, x, \lambda)\}, and, in particular, possible choices of p can be p(t, x, \lambda) = |\lambda|^{\alpha(t, x)} and p(t, x, \lambda) = (1 + |\lambda|^2)^{\gamma(t, x)}, generalizing the previous stationary in time families. Similar examples can be considered for p2. In this case, the class of admissible symbols p2 is wider than in the first case, since we can assume on spatial process X suitable second-order regularity and moment conditions. Generalized classes of heterogeneous autoregressive Hilbertian processes can be also introduced in this framework, thus extending the usual class of autocorrelation operators in terms of variable order pseudodifferential operators.

3 Estimation procedure

We briefly describe the elements of the multiple-window atomic decomposition involved in the estimation procedure proposed. Let us consider the C^\infty compactly supported functions k^0 and \hat{k}_0, with \hat{k}_0(0) \neq 0 and \hat{\hat{k}}_0(0) \neq 0, to define a resolution of the identity in the following way: For i = 1, \ldots, L, and for x \in \mathbb{R}^n, t > 0, and f \in \mathcal{S}'(\mathbb{R}^n),

k_{N_i}(y) = \Delta^{N_i}k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} \right)^{N_i} k^0(y), \text{ with } N_i \in \mathbb{N} \setminus \{0\}, i = 1, \ldots, L.

For each fixed t, the spatial local mean of process Y at scale a is then given by

k_{N_i}(t, a, Y)(x) = \int_{\mathbb{R}^n} k_{N_i}(y)Y(t, x+ay)dy = a^{-n} \int_{\mathbb{R}^n} k_{N_i} \left(\frac{y-x}{a}\right) Y(t, y)dy.

A particular example of the above defined local means is the multiple-window wavelet transform. The L associated families of atoms, involved in the atomic decomposition of process Y, are then defined as

A_{t,a,m}^{(i)}(x) = \lambda_{t,a,m}^{-1} \psi_{a,m}(x)k_{N_i}(t, a, Y)(x), \quad \forall x \in \mathbb{R}^n,

where

\lambda_{t,a,m}^{(i)} = a^{-\inf_{x \in Q_{a,m}} H_i(x)} \sup_{x \in Q_{a,m}} |k_{N_i}(t, a, Y)(x)|,

with \{\psi_{a,m}\}_{a \in \Lambda, m \in \mathbb{M}^n} defining at each scale a a resolution of unity, with compact supports \{Q_{a,m}\}_{a \in \Lambda, m \in \mathbb{M}^n} providing a partition of \mathbb{R}^n. Usually, \mathbb{M}^n = \mathbb{Z}^n and scale a is discretized considering a = 2^{-j}, j \in \mathbb{Z}. Since, for i = 1, \ldots, L,

E|A_{t,a,m}^{(i)}(x)|^2 = C_{t,\psi,a^{2H_i}(x)}, \quad a \to 0,
the estimation procedure proposed consists of computing an approximation of the above quantity based on

$$\frac{1}{L} \sum_{i=1}^{L} |A_{i,a,m}(x)|^2,$$

under suitable decorrelation conditions on the atoms.

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References


