Stability of Periodic Motions in Satellite Dynamics
Stability Theory for Hamiltonian Systems

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Course of Computer Algebra and Differential Equations
Contents

1. Stability of Nonautonomous Hamiltonian Systems
   - Linear Problem of Stability
   - Nonlinear Problem of Stability

2. Constructive algorithm for the normalization of a periodic Hamiltonian
   - Construction of the map
   - Normalization of the area preserving map
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Consider a system of differential equation

$$\frac{dx}{dt} = A(t)x, \quad A(t + 2\pi) = A(t)$$

(1)

**Theorem.** Let $X(t)$ be a fundamental matrix of solution of system (1) satisfying the following initial conditions $X(0) = E$. Then, for all $t \in \mathbb{R}$

$$X(t) = Y(t)e^{Bt}$$

where $B$ is a constant matrix and $Y(t)$ is a $2\pi$–periodic in $t$ matrix.
Lyapunov-Floquet theorem

**Theorem.** There exists a linear $2\pi$–periodic change of variables that transforms the $2\pi$–periodic linear system to an autonomous linear system.

Let us perform linear transformation

$$x = X(t)e^{-Bt}y$$

where $X(t)$ is the fundamental matrix of solution of system (1) defined above.

In the new variables we have

$$\frac{dy}{dt} = By. \tag{2}$$

The stability problems for systems (1) and (2) are equivalent.
The eigenvalues $\rho_i$ of matrix $X(2\pi)$ are called the characteristic multipliers of the system.

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$$

They are also the eigenvalues of the (linear) Poincare maps. The eigenvalues $\lambda_i$ of matrix $\mathbf{B}$ are called a characteristic exponent (sometimes called a Floquet exponent). The relation

$$\rho_i = e^{2\pi \lambda_i}$$

and takes place.
Linear Hamiltonian system

**Poincare – Lyapunov Theorem.** The characteristic equation

\[ \det(X(2\pi) - \rho) = 0 \]

for linear Hamiltonian system is a reciprocal equation.

- If linear Hamiltonian system has characteristic multiplier \( \rho_i \) then it also has the multiplier \( \rho_i^{-1} \).
- Linear periodic Hamiltonian system is stable if and only if all characteristic multipliers lie on unit circle and Jordan normal form of the matrix \( X(2\pi) \) is diagonal.
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Let us consider nonlinear $2\pi$–periodic Hamiltonian system with $n$ degrees of freedom. By using $2\pi$–periodic transformation its Hamiltonian can be brought into the following normal form

$$H = \omega_1 r_1 + \cdots + \omega_1 r_n + \sum_{i,j=1}^{n} a_{ij} r_i r_j + H^{(5)}(r, \varphi, t)$$

Let us introduce 'additional' pair of canonical variables $r_{n+1}$ and $\varphi_{n+1} = t$ and consider autonomous Hamiltonian system with $n + 1$ degrees of freedom. The Hamiltonian reads

$$H^* = \omega_1 r_1 + \cdots + \omega_1 r_n + r_{n+1} + \sum_{i,j=1}^{n} a_{ij} r_i r_j + H^{(5)}(r, \varphi, \varphi_{n+1})$$
Stability for most of initial conditions

If the Hamiltonian does not satisfy the condition of nondegeneracy

$$D_{k+1} = \det \left( \frac{\partial^2 H_0^*}{\partial r^2_*} \right) \equiv 0$$  \hspace{1cm} (3)

Let us notice that the condition of isoenergetic nondegeneracy never fulfilled, i.e

$$D_{k+2} = \det \begin{bmatrix} \left( \frac{\partial^2 H_0^*}{\partial r^2_*} \right) & \left( \frac{\partial H_0^*}{\partial r_*} \right) \\ \left( \frac{\partial H_0^*}{\partial r_*} \right) & 0 \end{bmatrix} \neq 0, \Rightarrow \quad D_k = \det \left( \frac{\partial^2 H_0}{\partial r^2} \right) \neq 0$$
Arnold diffusion in resonant cases
Markeev’s Example

Let us consider the system with Hamiltonian

\[ H = \omega_1 r_1 + \omega_2 r_2 - 24r_1^2 + 2r_1 r_2 + r_2^2 + H^{(1)}(r, \varphi, t), \]

where

\[ H^{(1)}(r, \varphi, t) = r_2^2 \sqrt{r_1} \sin(\varphi_1 + 4\varphi_2 - Nt). \]

and resonance \( \omega_1 + 4\omega_2 = N \) takes place.

The conditions of Arnold’s theorem are fulfilled

\[ D_3 = a_{12}^2 - a_{11} a_{22} = 25 \neq 0, \]

but the Hamiltonian system has the following solution

\[ \varphi_1 + 4\varphi_2 - Nt = \pi \]

\[ r_1(t) = \frac{1}{4} r_2(t) = r_1(0)[1 - 24r_3^2(0)t]^{2/3}. \]
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Formulation of the problem

Consider $2\pi$–periodic Hamiltonian system with two degrees of freedom with Hamiltonian

$$H(q_1, q_2, p_1, p_2, t) = H_2 + H_3 + H_4 \ldots$$

where $H_k$ is a form of degree $k$ with $2\pi$–periodic coefficients. Notations:

1. $q_i^{(0)}$ and $p_i^{(0)}$ are initial values of variables $q_i$ and $p_i$
2. $q_i^{(1)}$ and $p_i^{(1)}$ are values of variables $q_i$ and $p_i$ at $t = 2\pi$

If $q_i^{(0)}$ and $p_i^{(0)}$ are small enough, then $q_i^{(1)}$ and $p_i^{(1)}$ are analytic functions of $q_i^{(0)}$ and $p_i^{(0)}$. These functions define a map $T$ of the neighbourhood of equilibrium onto itself.
Let $X(t)$ be the fundamental matrix of the linear canonical system with Hamiltonian $H_2$. Its elements satisfy the equations

$$
\frac{dx_{js}}{dt} = \frac{\partial H_2}{\partial x_{j+2,s}}, \quad \frac{dx_{j+2,s}}{dt} = -\frac{\partial H_2}{\partial x_{js}},
$$

$$
H_2 = H_2(x_{1s}, x_{2s}, x_{2s}, x_{2s}, t) \quad (j = 1, 2; \quad s = 1, 2, 3, 4)
$$

and the initial conditions

$$
X(0) = E
$$
Linear transformation

Let us introduce new canonical variables

$$
\begin{pmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2
\end{pmatrix} = X(t)
\begin{pmatrix}
u_1 \\
u_2 \\
v_1 \\
v_2
\end{pmatrix}
$$

(5)

New Hamiltonian $G(u_1, u_2, v_1, v_2, t)$ does not contain quadratic terms in $u_1, u_2, v_1, v_2$:

$$G = G_3 + G_4 \ldots$$
Note that

\[ q_1^{(0)} = u_1^{(0)}, \ldots, p_2^{(0)} = v_2^{(0)}. \]

Thus we have to look for the map

\[ q_i^{(0)}, p_i^{(0)} \rightarrow u_i^{(1)}, v_i^{(1)}. \]

\[ q_i^{(0)} = \frac{\partial S}{\partial p_i^{(0)}}, \quad v_i^{(1)} = \frac{\partial S}{\partial u_i^{(1)}} \]

where

\[ S = u_1^{(1)} p_1^{(0)} + u_2^{(1)} p_2^{(0)} + S_3 \left( u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)} \right) + \ldots \]
In fact

\[ S\left( u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)} \right) = \Phi \left( u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, 2\pi \right) \]

where \( \Phi \left( u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)}, t \right) \) satisfies the Hamilton-Jacobi equation

\[ \frac{\partial \Phi}{\partial t} + G \left( u_1^{(1)}, u_2^{(1)}, \frac{\partial \Phi}{\partial u_1^{(1)}}, \frac{\partial \Phi}{\partial u_2^{(1)}}, t \right) = 0 \]
Construction of the generating function

We expand $\Phi$ in power series in $u_i^{(1)}, p_i^{(1)}$ ($i = 1, 2$)

$$\Phi = \Phi_3 + \Phi_4 + \ldots$$

From the Hamilton-Jacobi equation we have

$$\frac{\partial \Phi_3}{\partial t} = -G_3, \quad \frac{\partial \Phi_4}{\partial t} = -G_4 - \sum_{i=1}^{2} \frac{\partial G_3}{\partial p_i^{(0)}} \cdot \frac{\partial \Phi_3}{\partial u_i^{(1)}}, \quad \ldots$$

From equation (6) we calculate coefficients of forms $\Phi_k$ as function of $t$. 
Construction of the generating function

For example, coefficients of

\[ \Phi_3 = \sum_{i_1+i_2+j_1+j_2 = 3} \varphi_{i_1i_2j_1j_2} u_1^{(1)}, u_2^{(1)}, p_1^{(0)}, p_2^{(0)} \]

satisfy the following differential equation

\[ \frac{d\varphi_{i_1i_2j_1j_2}}{dt} = -g_{i_1i_2j_1j_2}(t) \]

are function of \( x_{ij} \). Initial conditions

\[ \varphi_{i_1i_2j_1j_2}(0) = 0 \]

Coefficients \( g_{i_1i_2j_1j_2} \) depend on elements \( x_{ij}(t) \) of fundamental matrix of linear system.
Thus, in order to calculate coefficients

\[ s_{i_1i_2j_1j_2} = \varphi_{i_1i_2j_1j_2}(2\pi) \]

of the form \( S_3 \).

We have integrate the following system

\[
\frac{dx_{js}}{dt} = \frac{\partial H_2}{\partial x_{j+2,s}}, \quad \frac{dx_{j+2,s}}{dt} = -\frac{\partial H_2}{\partial x_{js}} \quad (j = 1, 2; \quad s = 1, 2, 3, 4),
\]

\[ d\varphi_{i_1i_2j_1j_2} = -g_{i_1i_2j_1j_2}(t) \quad (i_1 + i_2 + j_1 + j_2 = 3) \]

with initial conditions

\[ x_{ij}(0) = 0 \quad (i \neq j) \quad x_{ij}(0) = 1 \quad (i = j) \quad \varphi_{i_1i_2j_1j_2}(0) = 0. \]

for interval from \( t = 0 \) to \( t = 2\pi \).
In order to calculate the map up to terms:
of the second order we solve $16 + 20 = 36$ equations
of the third order we solve $16 + 20 + 35 = 71$ equations.
Form of the area-preserving map

The area-preserving map reads

$$
\begin{pmatrix}
q_1^{(1)} \\
q_2^{(1)} \\
p_1^{(1)} \\
p_2^{(1)}
\end{pmatrix}
= \mathbf{X}(2\pi)
\begin{pmatrix}
\tilde{q}_1 \\
\tilde{q}_2 \\
\tilde{p}_1 \\
\tilde{p}_2
\end{pmatrix}
$$

Where

$$
\tilde{q}_j = q_j^{(0)} - \frac{\partial S_3}{\partial p_j^{(0)}} + \sum_{k=1}^{2} \frac{\partial^2 S_3}{\partial p_j^{(0)} \partial q_k^{(0)}} \cdot \frac{\partial S_3}{\partial p_k^{(0)}} - \frac{\partial S_4}{\partial q_j} + O_4
$$

$$
\tilde{p}_j = p_j^{(0)} + \frac{\partial S_3}{\partial q_j^{(0)}} - \sum_{k=1}^{2} \frac{\partial^2 S_3}{\partial q_j^{(0)} \partial q_k^{(0)}} \cdot \frac{\partial S_3}{\partial p_k^{(0)}} + \frac{\partial S_4}{\partial q_j} + O_4
$$
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Linear normalization of the map

\[
\begin{bmatrix}
q_1 \\
q_2 \\
p_1 \\
p_2 \\
Q_1 \\
Q_2 \\
P_1 \\
P_2
\end{bmatrix} = N
\left|
\begin{bmatrix}
Q_1 \\
Q_2 \\
P_1 \\
P_2
\end{bmatrix}
\right|
\]

(7)

In new variables the map reads

\[
Q_{j}^{(1)} = \rho_j \left( Q_{j}^{(0)} - \frac{\partial W_3}{\partial p_j^{(0)}} + \ldots \right)
\]

\[
P_{j}^{(1)} = \rho_{j+2} \left( P_{j}^{(0)} + \frac{\partial W_3}{\partial Q_j^{(0)}} + \ldots \right)
\]

\[
\rho_j = e^{i2\pi\sigma_j} \quad \rho_{j+2} = e^{-i2\pi\sigma_j}
\]

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Nonresonant case

Nonlinear close to identity transformation

\[ Q_1, Q_2, P_1, P_2 \rightarrow \xi_1, \xi_2, \eta_1, \eta_2 \]

In new variables:

\[ \xi_j^{(1)} = \rho_j \left( \xi_j^{(0)} - \frac{\partial Z_4}{\partial \eta_j^{(0)}} + \ldots \right) \]

\[ \eta_j^{(1)} = \rho_{j+2} \left( \eta_j^{(0)} + \frac{\partial Z_4}{\partial \eta_j^{(0)}} + \ldots \right) \]

\[ Z_4 = W_{2020} \xi_1^{(0)^2} \eta_1^{(0)^2} + W_{1111} \xi_1^{(0)} \xi_2^{(0)} \eta_1^{(0)} \eta_2^{(0)} + W_{0202} \xi_2^{(0)^2} \eta_2^{(0)^2} \]
Normal form of Hamiltonian

\[ H = i\sigma_1 \xi_1 \eta_1 + i\sigma_2 \xi_2 \eta_2 - \frac{1}{2\pi} (w_{2020} \xi_1^2 \eta_1^2 + \]
\[ + w_{111} \xi_1 \xi_2 \eta_1 \eta_2 + w_{0202} \xi_2 \eta_2 ) + O_5 \]

In canonical polar coordinates

\[ H = \sigma_1 r_1 + \sigma_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + O_5 \]

where

\[ c_{20} = \frac{1}{2\pi} w_{2020}, \quad c_{11} = \frac{1}{2\pi} w_{1111}, \]
\[ c_{02} = \frac{1}{2\pi} w_{0202} \]