Poisson and symplectic reductions of 4-D isotropic oscillators.
Application to the generalized van der Waals family

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Abstract

• We search for relative equilibria and their bifurcations on Hamiltonian system, by mean of normalization and reduction.

• Poisson and symplectic formalisms are compared.

• Poisson procedure relies on the quadratic invariants associated to the symmetries.

• The symplectic reduction is carried out a la Delaunay, hinging on the maximally superintegrable character of the isotropic oscillator.

• An application is performed on the generalized 4-DOF Van der Waals family of oscillators, includes models for perturbed Keplerian systems.
The problem

• We consider Hamiltonian systems

\[ q = \frac{\partial H}{\partial Q}, \quad Q = -\frac{\partial H}{\partial q} \]  

(1)

defined by perturbed Harmonic oscillators

\[ H = \frac{1}{2} \sum_{i}^{4} (Q_{i}^2 + \omega_{i} q_{i}^2) + \epsilon P(q, Q), \]  

(2)

where \( \epsilon \ll 1 \) is a parameter and \( P(q, Q) \) is the perturbation.

• We focus on the resonance 1:1:1:1:1. The quadratic part reads

\[ H_2 = H_\omega = \frac{1}{2} \sum_{i}^{4} (Q_{i}^2 + \omega q_{i}^2). \]  

(3)

and defines a 4-DOF system called isotropic oscillator.
4-D van der Waals family as benchmark

We will consider the uniparametric family of Hamiltonian systems defined by

$$\mathcal{H}_\beta(Q, q) = \mathcal{H}_2 + \varepsilon \mathcal{H}_6, \quad \varepsilon \ll 1,$$

where $\mathcal{H}_2$ is the isotropic oscillator and

$$\mathcal{H}_6(Q, q) = (q_1^2 + q_2^2 + q_3^2 + q_4^2) \left( \beta^2 (q_1^2 + q_2^2 - q_3^2 - q_4^2)^2 ight. + 4 (q_1^2 + q_2^2) (q_3^2 + q_4^2) \left. \right)$$

The system defined by $\mathcal{H}_\beta$ has two first integrals in involution

$$\Xi = q_1 Q_2 - Q_1 q_2 + q_3 Q_4 - Q_3 q_4,$$

$$L_1 = q_3 Q_4 - Q_3 q_4 - q_1 Q_2 + Q_1 q_2,$$

(4)
References


References II


- **Ferrer, S.**, On the connection of 4-D isotropic oscillator and Kepler systems, submitted to *J. Geometric Mechanics*. 

Departamento de Matemática Aplicada.
Normalization and Toral Reduction

Reduction to $\mathbb{CP}^3$

Proposition 1.- Given the action associated to the uniparametric group defined by $X_{H_2}$ we have that the algebra of polynomial invariants under that action is generated by

\[
\begin{align*}
\pi_1 &= Q_1^2 + q_1^2, & \pi_2 &= Q_2^2 + q_2^2, & \pi_3 &= Q_3^2 + q_3^2, \\
\pi_4 &= Q_4^2 + q_4^2, \\
\pi_5 &= Q_1 Q_2 + q_1 q_2, & \pi_6 &= Q_1 Q_3 + q_1 q_3, & \pi_7 &= Q_1 Q_4 + q_1 q_4, \\
\pi_8 &= Q_2 Q_3 + q_2 q_3, & \pi_9 &= Q_2 Q_4 + q_2 q_4, & \pi_{10} &= Q_3 Q_4 + q_3 q_4, \\
\pi_{11} &= -Q_1 q_2 + q_1 Q_2, & \pi_{12} &= -Q_1 q_3 + q_1 Q_3, & \pi_{13} &= -Q_1 q_4 + q_1 Q_4, \\
\pi_{14} &= -Q_2 q_3 + q_2 Q_3, & \pi_{15} &= -Q_2 q_4 + q_2 Q_4, & \pi_{16} &= -Q_3 q_4 + q_3 Q_4
\end{align*}
\]
• The invariants are obtained using canonical complex variables (see Egea 2007 for details).

• Normal form up to first order in those invariants

\[ \overline{\mathcal{H}} = \mathcal{H}_2 + \varepsilon \overline{\mathcal{H}}_6 \]  

where

\[ \mathcal{H}_2 = \frac{1}{2} (\pi_1 + \pi_2 + \pi_3 + \pi_4) = n \]  

and

\[ \overline{\mathcal{H}}_6 = \frac{1}{2} \left[ (1 - 4\beta^2) n (\pi_{15}^2 + \pi_{14}^2 + \pi_{13}^2 + \pi_{12}^2) 
+ 2(\beta^2 - 1) (\pi_{11}^2 (\pi_4 + \pi_3) - \pi_{16}^2 (\pi_3 + \pi_4)) 
+ 5(1 - \beta^2) n (\pi_9^2 + \pi_8^2 + \pi_7^2 + \pi_6^2) 
+ \beta^2 n (5n^2 - 3\pi_{11}^2) + (\beta^2 - 4) n \pi_{16}^2 \right] \]
• The reduction is now performed using the orbit map

\[ \rho_\pi : \mathbb{R}^8 \rightarrow \mathbb{R}^{16}; (q, Q) \rightarrow (\pi_1, \cdots, \pi_{16}) . \]

The image of this map is the orbit space for the \( H_2 \)-action. The image of the level surfaces \( H_2(q, Q) = n \) under \( \rho_\pi \) are the reduced phase spaces.

• These reduced phase spaces are isomorphic to \( \mathbb{C}P^3 \). The normalized Hamiltonian can be expressed in the invariants and therefore naturally lifts to a function on \( \mathbb{R}^{16} \), which, on the reduced phase spaces, restricts to the reduced Hamiltonian.
• In the following we will not use the invariants $\pi_i$, but instead use the $(K_i, L_j, J_k)$ invariants, a linear coordinate transformation on the image of the orbit map.
• By this change of coordinates *the first integrals* are now among the invariants defining the image.

\[
\begin{align*}
H_2 &= \frac{1}{2} (\pi_1 + \pi_2 + \pi_3 + \pi_4) \\
L_2 &= \pi_{12} + \pi_{15} \\
K_1 &= \frac{1}{2} (-\pi_1 - \pi_2 + \pi_3 + \pi_4) \\
J_7 &= \pi_{12} - \pi_{15} \\
J_1 &= \frac{1}{2} (\pi_1 - \pi_2 - \pi_3 + \pi_4) \\
L_3 &= \pi_{14} - \pi_{13} \\
J_2 &= \frac{1}{2} (\pi_1 - \pi_2 + \pi_3 - \pi_4) \\
J_8 &= \pi_{14} + \pi_{13} \\
K_2 &= \pi_8 - \pi_7 \\
K_3 &= -\pi_6 - \pi_9 \\
J_3 &= \pi_8 + \pi_7 \\
J_6 &= \pi_6 - \pi_9 \\
J_4 &= \pi_5 + \pi_{10} \\
\Xi &= \pi_{16} + \pi_{11} \\
J_5 &= \pi_5 - \pi_{10} \\
L_1 &= \pi_{16} - \pi_{11}
\end{align*}
\]
The normal form in these invariants is

\[ \overline{H}_\Xi = \frac{1}{2} \left[ n \left( 5 K_2^2 + 5 K_3^2 + 2 L_1^2 + L_2^2 + L_3^2 ight) + \beta^2 \left( 5 K_1^2 + L_2^2 + L_3^2 \right) \right] - \left( (4 + \beta^2) (K_2 L_2 + K_3 L_3) + (2 + 3 \beta^2) K_1 L_1 \right) \xi \]

The reduction of the \( H_2 \) action may now be performed through the orbit map

\[ \rho_{K,L,J} : \mathbb{R}^8 \rightarrow \mathbb{R}^{16}; (q, Q) \rightarrow (H_2, \cdots, J_8). \]

Note that on the orbit space we have the reduced symmetries due to the reduced actions given by the reduced flows of \( X_\Xi \) and \( X_{L_1} \).
Toroidal reduction over $\mathbb{C}P^3$. Rotacional invariants

Reduction by $\Xi = \xi$. The second reduced space

$$S^2_{n+\xi} \times S^2_{n-\xi}$$

Proposition.- Let us $\rho$ be the $S^1$-action generated by the Poisson flow of $\Xi$ over $\mathbb{C}P^3$. The functions

$$H_2 = \frac{1}{2} (\pi_1 + \pi_2 + \pi_3 + \pi_4),$$
$$L_1 = -\pi_{11} + \pi_{16},$$
$$L_3 = -\pi_{13} + \pi_{14},$$
$$K_2 = -\pi_7 + \pi_8,$$

are $\rho$-invariants over $\mathbb{C}P^3$. 

$$\Xi = \pi_{11} + \pi_{16},$$
$$L_2 = \pi_{12} + \pi_{15},$$
$$K_1 = \frac{1}{2} (-\pi_1 - \pi_2 + \pi_3 + \pi_4),$$
$$K_3 = -\pi_6 - \pi_9.$$
This, in turn, leads us to the orbit mapping

\[ \rho_2 : \mathbb{R}^{16} \to \mathbb{R}^8; (\pi_1, \cdots, \pi_{16}) \to (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi). \]

The orbit space \( \rho_2(\mathbb{C}P^3) \) is defined as a six dimensional algebraic variety in \( \mathbb{R}^8 \) by the two relations

\[ K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 = H_2^2 + \Xi^2, \tag{9} \]

\[ K_1L_1 + K_2L_2 + K_3L_3 = H_2\Xi. \]

The reduced phase spaces are obtained by setting

\[ H_2 = n, \quad \Xi = \xi. \]
Second reduced space is isomorphic to $S^2_{n+\xi} \times S^2_{n-\xi}$. This is shown introducing the change of coordinates

$$
\sigma_1 = K_1 + L_1, \quad \sigma_2 = K_2 + L_2, \quad \sigma_3 = K_3 + L_3,
\delta_1 = L_1 - K_1, \quad \delta_2 = L_2 - K_2, \quad \delta_3 = L_3 - K_3.
$$

Applied to (9) we obtain

$$
\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = (n + \xi)^2,
\delta_1^2 + \delta_2^2 + \delta_3^2 = (n - \xi)^2,
$$

in other words $S^2_{n+\xi} \times S^2_{n-\xi}$. When $\xi = 0$, it corresponds to the first reduced space of Keplerian systems by the energy.
The problem References Invariants, symmetries and reduced orbit spaces Symplectic reduction Delaunay Normalization Searching for relative equilibria

Figure: Double reduced spaces $S^2_{n+\xi} \times S^2_{n-\xi}$ for different values of the integral $\xi$
The second reduced Hamiltonian up to first order, modulo a constant takes the form

\[
\overline{H}_\Xi = \frac{1}{2} \left[ n \left( 5 K_2^2 + 5 K_3^2 + 2 L_1^2 + L_2^2 + L_3^2 + \right. \\
+ \beta^2 \left( 5 K_1^2 + L_2^2 + L_3^2 \right) \right] \\
- \left( (4 + \beta^2) (K_2 L_2 + K_3 L_3) + (2 + 3 \beta^2) K_1 L_1 \right) \xi
\]
Reduction by \( L_1 = \ell \). Thrice reduced space \( V_{n, \xi, \ell} \)

**Proposition.**- Let \( \rho_2 \) be the \( S^1 \)-action generated by the Poisson flow defined by \( L_1 = \pi_{16} - \pi_{11} \) over \( S^2_{n+\xi} \times S^2_{n-\xi} \). The functions

\[
M = \frac{1}{2} (K_2^2 + K_3^2 + L_2^2 + L_3^2), \quad N = \frac{1}{2} (K_2^2 + K_3^2 - L_2^2 - L_3^2),
\]

\[
Z = K_2L_2 + K_3L_3, \quad S = K_2L_3 - K_3L_2, \quad K = K_1.
\]

generate the algebra of \( \rho_2 \)-invariants functions constrained by the relations

\[
K^2 + L_1^2 + 2M = n^2 + \xi^2,
\]

\[
KL_1 + Z = n\xi, \tag{12}
\]

\[
M^2 - N^2 = Z^2 + S^2.
\]
Then, we have defined the orbit map

$$\rho_2 : \mathbb{R}^6 \to \mathbb{R}^6; \ (K_1, K_2, K_3, L_1, L_2, L_3) \to (M, N, Z, S, K, L_1).$$

When we fix a value of $L_1 = \ell$, then the relations (12) define the \textit{thrice reduced space}

$$V_{n\xi\ell} = \{ (K, S, N) \mid N^2 + S^2 = f(K),$$

$$f(K) = ((n + \xi)^2 - (K + \ell)^2)((n - \xi)^2 - (K - \ell)^2) \}$$

which is a surface of revolution, obtained rotating $\sqrt{f(K)}$ around the axis $K$. 
\[ f(K) = (K + n + \xi + \ell)(K - n - \xi + \ell)(K - n + \xi - \ell)(K + n - \xi - \ell), \]

The roots are given by

\[ K_1 = -\ell - n - \xi, \quad K_2 = \ell + n - \xi, \quad K_3 = \ell - n + \xi, \quad K_4 = -\ell + n + \xi. \]

So \( f(K) \) is positive (or zero) in the subsequent intervals of \( K \):

\[
\begin{align*}
  l < \xi, & \quad -l < \xi & \quad K_1 < K_3 < K_2 < K_4 & \quad K \in [K_3, K_2] \\
  l > \xi, & \quad -l < \xi & \quad K_1 < K_3 < K_4 < K_2 & \quad K \in [K_3, K_4] \\
  l < \xi, & \quad -l > \xi & \quad K_3 < K_1 < K_2 < K_4 & \quad K \in [K_1, K_2] \\
  l > \xi, & \quad -l > \xi & \quad K_3 < K_1 < K_4 < K_2 & \quad K \in [K_1, K_4]
\end{align*}
\] (13)
Thrice reduced spaces $V_n \xi \ell$

**Figure:** Thrice reduced space over the space of integrals. The vector $(K, N, S)$ represents the coordinates.
The Hamiltonian on the third reduced phase space is

\[
\overline{\mathcal{H}}_{\Xi, L_1} = \frac{3n}{4} (3\lambda - 2) K^2 + \xi l (1 - \lambda) K + \frac{n}{2} (4 - \lambda) N \\
+ n^3 \left( \frac{3}{2} + \frac{\lambda}{4} \right) - (l^2 + \xi^2) \left( \frac{\lambda}{2} + 1 \right) \frac{n}{2}
\]

\( (14) \)

- In \((K, N, S)\)-space the energy surfaces are parabolic cylinders.
- The intersection with the reduced phase space give the trajectories of the reduced system.
- Tangency with the reduce phase spaces gives relative equilibria that generically will correspond to three dimensional tori in the original phase space.
Bifurcations of the polar vdWaals family
Bifurcations of the polar vdWaals family

Figure: Evolution of the reduced flow with the physical parameter $\lambda$. 
Symplectic charts for isotropic oscillators

The goal now is to express the isotropic oscillator in a convenient form to implement Lie transforms.

- 1:1
  - Complex variables; Cushman, Rod, . . .
  - Lissajous variables; Deprit
- 1:1:1
  - Lissajous 3-DOF; Ferrer, Gárate, Yanguas, Palacián, . . .
- 1:1:. . . :1
  - Lissajous n-DOF; Deprit and Elipe
- 1:1:1:1
  - Lissajous 4-DOF; Deprit and Elipe
  - Delaunay 4-DOF; Ferrer
4-DOF Lissajous variables

\[
(\psi_1, \psi_2, \psi_3, \psi_4, \Psi_1, \Psi_2, \Psi_3, \Psi_4) \rightarrow (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4)
\]

\[
q_1 = \sqrt{-\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4} \omega \sin(\psi_1 + \psi_2 + \psi_3 + \psi_4),
\]

\[
q_2 = \sqrt{\Psi_1 - \Psi_2} \omega \sin(\psi_1 - \psi_2 + \psi_3 + \psi_4),
\]

\[
q_3 = \sqrt{\Psi_1 - \Psi_3} \omega \sin(\psi_1 + \psi_2 - \psi_3 + \psi_4),
\]

\[
q_4 = \sqrt{\Psi_1 - \Psi_4} \omega \sin(\psi_1 + \psi_2 + \psi_3 - \psi_4),
\]

\[
Q_1 = \sqrt{(-\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4)\omega \cos(\psi_1 + \psi_2 + \psi_3 + \psi_4)},
\]

\[
Q_2 = \sqrt{(\Psi_1 - \Psi_2)\omega \cos(\psi_1 - \psi_2 + \psi_3 + \psi_4)},
\]

\[
Q_3 = \sqrt{(\Psi_1 - \Psi_3)\omega \cos(\psi_1 + \psi_2 - \psi_3 + \psi_4)},
\]

\[
Q_4 = \sqrt{(\Psi_1 - \Psi_4)\omega \cos(\psi_1 + \psi_2 + \psi_3 - \psi_4)},
\]
Switching to symplectic reduction

\[
\begin{pmatrix}
q_1, & q_2, & q_3, & q_4 \\
Q_1, & Q_2, & Q_3, & Q_4
\end{pmatrix}
\xrightarrow{\text{Projective Euler}}
\begin{pmatrix}
\rho, & \phi, & \theta, & \psi \\
P, & \Phi, & \Theta, & \Psi
\end{pmatrix}
\xleftarrow{\text{Proj. Andoyer}}
\begin{pmatrix}
\ell, & \lambda, & g, & \nu \\
L, & \Lambda, & G, & N
\end{pmatrix}
\xrightarrow{\text{Delaunay}}
\begin{pmatrix}
\rho, & \lambda, & \mu, & \nu \\
P, & \Lambda, & M, & N
\end{pmatrix}
\]

Adding the regularization \( dt = 1/(4\rho) \, d\tau \), we end up with the Hamiltonian of the isotropic oscillator given by

\[
\mathcal{H}_\omega = -\frac{\gamma^2}{2L^2}
\]
4-D Delaunay normalization.

We implement the normalization of the variable $\ell$, by a Lie transform

$$T_W : (\ell, \lambda, g, \nu, L, \Lambda, G, N) \to (\ell', \lambda', g', \nu', L', \Lambda', G', N')$$

which is defined by the IVP

$$\frac{dx}{d\epsilon} = \frac{\partial W}{\partial X}, \quad \frac{dX}{d\epsilon} = -\frac{\partial W}{\partial x}, \quad (x, X)(0) = (x', X')$$

where the generating function $W$ takes the form

$$W = \sum_{i \geq 0} \frac{\epsilon^i}{i!} W_{i+1}.$$
• The Hamiltonian in the old variables is

\[ \mathcal{H} = -\frac{\gamma^2}{2L^2} + \sum_{i=1}^{k} \frac{\epsilon^i}{i!} \mathcal{P}(\ell, \lambda, g, \nu, L, \Lambda, G, N) \]

• The Hamiltonian in the new variables reads

\[ \mathcal{K} = -\frac{\gamma^2}{2L'^2} + \sum_{i=1}^{k} \frac{\epsilon^i}{i!} \mathcal{P}(-, \lambda, g', \nu', L', \Lambda', G', N') \]

\[ + \mathcal{O}^{(k+1)}(\ell', \gamma', g', \nu', L', \Lambda', G', N'), \]

which ‘relegates’ the variable \( \ell' \) up to order \( k + 1 \).
Application to the van der Waals family:

\[
\mathcal{K} = -\frac{\gamma^2}{2(L')^2} + \sum \frac{\epsilon^i}{i!} \mathcal{K}_i(-, g', -, -, L', G', \Lambda', N'; \beta) + \mathcal{O}(\epsilon^{i+1})
\]

\[
= \mathcal{K}_0 + \epsilon \mathcal{K}_1 + \frac{\epsilon^2}{2} \mathcal{K}_2 + \mathcal{O}(\epsilon^3)
\]

\[
= \mathcal{K}_0 + \epsilon \sum_{0}^{2} C_{i1} \cos i g + \frac{\epsilon^2}{2} \sum_{0}^{4} C_{i2} \cos i g + \mathcal{O}(\epsilon^3)
\]

where \( C_{ij} = C_{ij}(L, G, \Lambda, N; \beta) \).

\[
\mathcal{K}_1 = \frac{1}{2\pi} \int \mathcal{H}_1 \, d\ell = \frac{1}{2\pi} \int \sum (C_i \cos iE + S_i \sin iE)(1 - e \cos E) \, dE
\]

\[
= C_{01} + C_{11} \cos g + C_{21} \cos 2g.
\]

\[
C_{01} = \frac{1}{4} a^2 (2 + 3e^2)[2 + (\beta^2 - 1)(2c_1^2 c_2^2 + s_1^2 s_2^2)],
\]

\[
C_{11} = -3a^2 (\beta^2 - 1)(4 + e^2) e c_1 c_2 s_1 s_2,
\]

\[
C_{21} = \frac{5}{4} a^2 (\beta^2 - 1)e^2 s_1^2 s_2^2.
\]
\[ C_{02} = -\frac{a^5}{3072\gamma} \left( 480c^2 e^6 s^2 \alpha^2 \kappa^2 \sigma^2 - 8(96 + 96s^2 \alpha \sigma^2 + 96c^4 \alpha^2 \kappa^2 (\kappa^2 + \sigma^2) \right. \\
\left. \quad + s^4 \alpha^2 \sigma^2 (12\kappa^2 + 5\sigma^2) + 12c^2 \alpha(8s^2 \alpha \kappa^4 + s^2 \alpha \sigma^4 + 4\kappa^2 (4 + 5s^2 \alpha \sigma^2)) \right) \\
\left. \quad + e^4 (504 + 504s^2 \alpha \sigma^2 + 8c^4 \alpha^2 \kappa^2 (63\kappa^2 - 92\sigma^2) + s^4 \alpha^2 \sigma^2 (-640\kappa^2 + 61\sigma^2) \right) \\
\left. \quad - 8c^2 \alpha(92s^2 \alpha \kappa^4 + 80s^2 \alpha \sigma^4 - \kappa^2 (126 + 131s^2 \alpha \sigma^2)) \right) \\
\left. \quad - 8e^2 (396 + 396s^2 \alpha \sigma^2 + 12c^4 \alpha^2 \kappa^2 (33\kappa^2 + 28\sigma^2) + s^4 \alpha^2 \sigma^2 (170\kappa^2 + 229\sigma^2) \\
\left. \quad \quad \quad \quad \quad \quad + 2c^2 \alpha(168s^2 \alpha \kappa^4 + 85s^2 \alpha \sigma^4 + \kappa^2 (396 + 226s^2 \alpha \sigma^2))) \right) \right), \\
\]

\[ C_{12} = +\frac{a^5 cs \alpha \kappa \sigma}{384\gamma} e \left( e^4 (192 + 4c^2 \alpha(58\kappa^2 - 9\sigma^2) + s^2 \alpha(-36\kappa^2 + 191\sigma^2)) \right) \\
\left. \quad - 2e^2 (354 + 2c^2 \alpha(69\kappa^2 + 145\sigma^2) + s^2 \alpha(290\kappa^2 + 157\sigma^2)) \right) \\
\left. \quad - 16(75 + 3c^2 \alpha(13\kappa^2 + 9\sigma^2) + s^2 \alpha(27\kappa^2 + 40\sigma^2)) \right), \\
\]

\[ C_{22} = -\alpha a^5 s^2 \sigma^2 \frac{e^2}{768\gamma} \left( 120c^2 e^4 \alpha \kappa^2 + e^2 (114 + 2c^2 \alpha(281\kappa^2 - 64\sigma^2) + s^2 \alpha(-128\kappa^2 + 57\sigma^2)) \right) \\
\left. \quad - 6(162 + 2c^2 \alpha(53\kappa^2 + 33\sigma^2) + s^2 \alpha(66\kappa^2 + 81\sigma^2)) \right) \right), \\
\]

\[ C_{32} = +\frac{5\alpha^2 a^5}{384\gamma} e^3 (2 + 17e^2) c \kappa s^3 \sigma^3, \\
\]

\[ C_{42} = -\frac{95\alpha^2 a^5}{3072\gamma} e^4 s^4 \sigma^4, \]
First order normalized diff system

Thus the differential system corresponding to the first order normalized Hamiltonian is

\[ \dot{\ell} = \frac{\partial H}{\partial L} = \frac{\gamma^2}{L^3} + \varepsilon \frac{\partial \mathcal{P}}{\partial L}, \quad \dot{L} = -\frac{\partial H}{\partial \ell} = 0, \]

\[ \dot{g} = \frac{\partial H}{\partial G} = \varepsilon \frac{\partial \mathcal{P}}{\partial G}, \quad \dot{G} = -\frac{\partial H}{\partial g}, \]

\[ \dot{\lambda} = \frac{\partial H}{\partial \Lambda} = \varepsilon \frac{\partial \mathcal{P}}{\partial \Lambda}, \quad \dot{\Lambda} = -\frac{\partial H}{\partial \lambda} = 0 \]

\[ \dot{\nu} = \frac{\partial H}{\partial N} = \varepsilon \frac{\partial \mathcal{P}}{\partial N}, \quad \dot{N} = -\frac{\partial H}{\partial \nu} = 0 \]

In other words \( L, \Lambda \) and \( N \) are integrals, as we already know.

NB: We simplify expressions dropping primes.
The differential system splits into a 1-DOF Hamiltonian system, namely

\[
\dot{g} = \frac{\partial H}{\partial G} = \varepsilon \frac{\partial P}{\partial G},
\]

\[
\dot{G} = -\frac{\partial H}{\partial g} = -\varepsilon \frac{\partial P}{\partial g},
\]

and three quadratures over the previous solved system

\[
\dot{\ell} = \frac{\partial H}{\partial L} = \frac{\gamma^2}{L^3} + \varepsilon \frac{\partial P}{\partial L},
\]

\[
\dot{\lambda} = \frac{\partial H}{\partial \Lambda} = \varepsilon \frac{\partial P}{\partial \Lambda},
\]

\[
\dot{\nu} = \frac{\partial H}{\partial N} = \varepsilon \frac{\partial P}{\partial N},
\]
Searching for relative equilibria and invariant tori

- $\mathbb{S}^1_{L,G,\Lambda,N}(\ell)$ periodic orbits

Solutions related to the unperturbed system are equivalent (Theorems of Reeb, Marsden, Weinstein) to find the roots of the reduced system

\[
\begin{align*}
\dot{g} &= \frac{\partial H}{\partial G} = \varepsilon \frac{\partial P}{\partial G} = 0, \\
\dot{G} &= -\varepsilon \frac{\partial P}{\partial g} = 0, \\
\dot{\lambda} &= \frac{\partial H}{\partial \Lambda} = \varepsilon \frac{\partial P}{\partial \Lambda} = 0, \\
\dot{\nu} &= \frac{\partial H}{\partial N} = \varepsilon \frac{\partial P}{\partial N} = 0,
\end{align*}
\]
• $\mathbb{T}^2_{L,G,\Lambda,N}(\ell, \nu)$ **Invariant 2-tori**

They correspond to the roots of the subsystem

\[
\dot{g} = \frac{\partial H}{\partial G} = \varepsilon \frac{\partial P}{\partial G} = 0, \quad \dot{G} = -\varepsilon \frac{\partial P}{\partial g} = 0,
\]

\[
\dot{\lambda} = \frac{\partial H}{\partial \Lambda} = \varepsilon \frac{\partial P}{\partial \Lambda} = 0,
\]

• $\mathbb{T}^2_{L,G,\Lambda,N}(\ell, \lambda)$ **Invariant 2-tori**

They correspond to the roots of the subsystem

\[
\dot{g} = \frac{\partial H}{\partial G} = \varepsilon \frac{\partial P}{\partial G} = 0, \quad \dot{G} = -\varepsilon \frac{\partial P}{\partial g} = 0,
\]

\[
\dot{\nu} = \frac{\partial H}{\partial N} = \varepsilon \frac{\partial P}{\partial N} = 0.
\]
• $T^3_{L,G,\Lambda,N}(\ell, \lambda, \nu)$ \textit{Invariant 3-tori}

They are related to the analysis of the thrice reduced system defined by

$$\dot{g} = \varepsilon \frac{\partial P}{\partial G} = 0,$$

$$\dot{G} = -\varepsilon \frac{\partial P}{\partial g} = 0,$$