Integrability analysis of a polynomial system of ODEs near a degenerate stationary point

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We consider an autonomous system of ordinary differential equations, which is solved with respect to derivatives.

To study local integrability of the system near a degenerate stationary point, we use an approach based on Power geometry and on the computation of the resonant normal form.

For the concrete planar 5-parametric system, we found the complete set of necessary conditions on parameters of the system for which the system is locally integrable near a degenerate stationary point.

This set consists of 4 two-parametric sets in this 5-parametric space. For 3 such sets we found by independent methods sufficient conditions of a local integrability.

Because these methods are constructive we get for these 3 sets first integrals of the system. So at these set of parameters the system is globally integrable.

For the forth set we have at the moment only approximations of the local integrals as truncated power series in parameters of the system, but we believe that it is possible to sum them up to finite functions.
Introduction

We consider an autonomous system of ordinary differential equations

$$\frac{dx_i}{dt} \overset{\text{def}}{=} \dot{x}_i = \varphi_i(X), \quad i = 1, \ldots, n,$$

(1)

where $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\varphi_i(X)$ are polynomials.

In a neighborhood of the stationary point $X = X^0$, the system (1) is locally integrable if it has there sufficient number $m$ of independent first integrals of the form

$$a_j(X)/b_j(X), \quad j = 1, \ldots, m,$$

where functions $a_j(X)$ and $b_j(X)$ are analytic in a neighborhood of the point $X = X^0$. Otherwise we call the system (1) locally nonintegrable in this neighborhood.
In [BrunoEdneral:2009], it was proposed a method of analysis of integrability of a system based on power transformations and computation of normal forms near stationary solutions of transformed systems [Bruno:1998].

In this report we demonstrate how this approach can be applied to the study of local integrability of the planar case (i.e. \( n = 2 \)) of the system (1) near the stationary point \( X^0 = 0 \) of high degeneracy.
In the neighborhood of the stationary point $X = 0$ the system (1) can be written in the form
\[ \dot{X} = A X + \tilde{\Phi}(X), \]
where $\tilde{\Phi}(X)$ has no linear in $X$ terms.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the matrix $A$. If at least one of them $\lambda_i \neq 0$, then the stationary point $X = 0$ is called an elementary stationary point. In this case the system (2) has a normal form which is equivalent to a system of lower order [Bruno:1979]. If all eigenvalues vanish, then the stationary point $X = 0$ is called a nonelementary stationary point. In this case there is no normal form for the system (2). But by using power transformations, a nonelementary stationary point $X = 0$ can be blown up to a set of elementary stationary points. After that, it is possible to compute the normal form and verify that the condition $A$ (see later) is satisfied [Bruno:1971] in each elementary stationary point.
If $n = 2$ then rationality of the ratio $\lambda_1/\lambda_2$ and the condition A (see the next paragraph) are necessary and sufficient conditions for local integrability of a system near an elementary stationary point. For local integrability of original system (1) near a degenerate (nonelementary) stationary point, it is necessary and sufficient to have local integrability near each of elementary stationary points, which are produced by the blowing up process described above.
So we study the system

\[
\begin{align*}
\dot{x}_1 &= x_1 \sum \phi_Q X^Q, \\
\dot{x}_2 &= x_2 \sum \psi_Q X^Q,
\end{align*}
\]

where \( Q = (q_1, q_2) \), \( X^Q = x_1^{q_1} x_2^{q_2} \); \( \phi_Q \) and \( \psi_Q \) are constant coefficients, which can be polynomials in parameters of the system.

\( Q \) is an integer vector with mainly nonnegative elements. Only a single element \( q_i \) can be equal -1.
About Normal Form and the Condition A

Let the linear transformation

$$X = BY$$  \hspace{1cm} (11)

bring the matrix $A$ to the Jordan form $J = B^{-1}AB$ and (2) to

$$\dot{Y} = JY + \tilde{\Phi}(Y).$$  \hspace{1cm} (12)

Let the formal change of coordinates

$$Y = Z + \Xi(Z),$$  \hspace{1cm} (13)

where $\Xi = (\xi_1, \ldots, \xi_n)$ and $\xi_j(Z)$ are formal power series, transform (12) in the system

$$\dot{Z} = JZ + \Psi(Z).$$  \hspace{1cm} (14)
We write it in the form
\[ \dot{z}_j = z_j g_i(Z) = z_j \sum g_j Q^Q \text{ over } Q \in \mathbb{N}_j, \ j = 1, \ldots, n, \] (15)
where \( Q = (q_1, \ldots, q_n), \ Z^Q = z_1^{q_1} \cdots z_n^{q_n}, \)
\[ \mathbb{N}_j = \{Q : Q \in \mathbb{Z}^n, \ Q + E_j \geq 0\}, \ j = 1, \ldots, n, \]
\( E_j \) means the unit vector. Denote
\[ \mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_n. \] (16)

The diagonal \( \Lambda = (\lambda_1, \ldots, \lambda_n) \) of \( J \) consists of eigenvalues of the matrix \( A. \)
System (14), (15) is called the resonant normal form if:

a) \( J \) is the Jordan matrix,

b) in writing (15), there are only the resonant terms, for which the scalar product
\[ \langle Q, \Lambda \rangle \overset{\text{def}}{=} q_1 \lambda_1 + \ldots + q_n \lambda_n = 0. \] (17)

**Theorem 2.1 (Bruno [4])** There exists a formal change (13) reducing (12) to its normal form (14), (15).
In [Bruno:1971] was proved that there are conditions on the normal form (15), which guarantee the convergence of the normalizing transformation (13).

**Condition A.** *In the normal form (15)*

\[ g_j = \lambda_j \alpha(Z) + \bar{\lambda}_j \beta(Z), \quad j = 1, \ldots, n, \]

where \( \alpha(Z) \) and \( \beta(Z) \) are some power series.

Let

\[ \omega_k = \min |\langle Q, \Lambda \rangle| \text{ over } Q \in \mathbb{N}, \quad \langle Q, \Lambda \rangle \neq 0, \quad \sum_{j=1}^{n} q_j < 2^k, \quad k = 1, 2, \ldots \]

**Condition \( \omega \) (on small divisors).** The series

\[ \sum_{k=1}^{\infty} 2^{-k} \log \omega_k > -\infty, \]

i.e. it converges.

It is fulfilled for almost all vectors \( \Lambda \).
Theorem 2.2 Bruno: 1971. If vector $A$ satisfies Condition $\omega$ and the normal form (2.6) satisfies Condition $A$ then the normalizing transformation (13) converges.

The algorithm of a calculation of the normal form, the normalizing transformation and the corresponding computer program are described in [Ednerald:2007].
The Simplest Nontrivial Example

We consider the system

\[
\begin{align*}
\frac{dx}{dt} &= -y^3 - bx^3y + a_0x^5 + a_1x^2y^2, \\
\frac{dy}{dt} &= (1/b)x^2y^2 + x^5 + b_0x^4y + b_1xy^3,
\end{align*}
\]

with arbitrary complex parameters \(a_i, b_i\) and \(b \neq 0\).

\[
\begin{align*}
\frac{dx}{dt} &= x(-x^{-1}y^3 - bx^2y + a_0x^4 + a_1xy^2) \\
\frac{dy}{dt} &= y((1/b)x^2y + x^5y^{-1} + b_0x^4 + b_1xy^2)
\end{align*}
\]
Systems with a nilpotent matrix of the linear part are thoroughly studied by Lyapunov et. al. In our example, there is no linear part, and the first approximation is not homogeneous but quasi homogeneous. This is the simplest case of a planar system without linear part with Newton’s open polygon consisting of a single edge. In our case the system corresponds to the quasi homogeneous first approximation with $R = (2, 3), \ s = 7$. In general case such problems have not been studied, and the authors do not know of any applications of the system (18).
System (3) has a quasi-homogeneous initial approximation if there exists an integer vector \( R = (r_1, r_2) > 0 \) and a number \( s \) such that the scalar product

\[
\langle Q, R \rangle \overset{\text{def}}{=} q_1 r_1 + q_2 r_2 \geq s = \text{const}
\]

for nonzero \(|\phi_Q| + |\psi_Q| \neq 0\), and between vectors \( Q \) with \( \langle Q, R \rangle = s \) there are vectors of the form \((q_1, -1)\) and \((-1, q_2)\). In this case, the system (3) takes the form

\[
\begin{align*}
\dot{x}_1 &= x_1[\phi_s(X) + \phi_{s+1}(X) + \phi_{s+2}(X) + \ldots], \\
\dot{x}_2 &= x_2[\psi_s(X) + \psi_{s+1}(X) + \psi_{s+2}(X) + \ldots],
\end{align*}
\]

where \( \phi_k(X) \) is the sum of terms \( \phi_Q X^Q \) for which \( \langle Q, R \rangle = k \). And the same holds for the \( \psi_k(X) \). Then the initial approximation of (3) is the quasi-homogeneous system

\[
\begin{align*}
\dot{x}_1 &= x_1 \phi_s(X), \\
\dot{x}_2 &= x_2 \psi_s(X).
\end{align*}
\]
We study the problem: what are the conditions on parameters under which the full system (18) is locally integrable. The local integrability of the first approximation (4) is necessary for this. For an autonomous planar system $m = 1$; so there are two cases:

1. System (4) is Hamiltonian, i.e. it has the form

$$\dot{x}_1 = \frac{\partial H(X)}{\partial x_2}, \quad \dot{x}_2 = -\frac{\partial H(X)}{\partial x_1},$$

where $H(X)$ is a quasi-homogeneous polynomial.

2. System (4) is non Hamiltonian, but it has the first integral $F(X)$:

$$\frac{\partial F(X)}{\partial x_1} x_1 \phi_3 + \frac{\partial F(X)}{\partial x_2} x_2 \psi_3 = 0,$$

where $F(X)$ is a quasi-homogeneous polynomial.
For the first case, with the additional assumption that the polynomial $H(X)$ is expandable into the product of only square free factors, the problem is solved in [Algaba et.al.:2009]. Therefore here we discuss only the second case. More precisely, we study the system with $R=(2,3)$ and $s=7$.

At $R = (2, 3)$ and $s = 7$ the quasi-homogeneous system (4) has the form

$$\dot{x} = ay^3 + bx^3 y, \quad \dot{y} = cx^2 y^2 + dx^5,$$

where $a \neq 0$ and $d \neq 0$.

$$\frac{dx}{dt} = x(ax^{-1}y^3 + bx^2 y)$$

$$\frac{dy}{dt} = y(cx^2 y + dx^5 y^{-1})$$
Lemma 1.1 If the system (5) with \( b \neq 0 \) and \( c \neq 0 \) has the first integral

\[
I = \alpha y^4 + \beta x^3 y^2 + \gamma x^6, \quad \beta \neq 0,
\]

then

\[
(\alpha d - b c)(3b + 2c) = 0.
\]
Proof. A derivative of the integral (6) with respect to the system (5) has the form
\[
\frac{\partial I}{\partial x}(ay^3 + bx^3y) + \frac{\partial I}{\partial y}(cx^2y^2 + dx^5) =
\]
\[
= (3\beta a + 4\alpha c)x^2y^5 + (6\gamma a + 3\beta b + 2\beta c + 4\alpha d)x^5y^3 +
\]
\[
+ (6\gamma b + 2\beta d)x^8y \equiv 0,
\]
thus coefficients at three monomials \(x^py^q\) are equal to zero, i.e.
\[
3\beta a + 4\alpha c = 0, \quad 6\gamma b + 2\beta d = 0,
\]
\[
6\gamma a + 3\beta b + 2\beta c + 4\alpha d = 0. \tag{8}
\]
From the first two equations (8), we obtain
\[
\alpha = -\frac{3\beta a}{4c}, \quad \gamma = -\frac{\beta d}{3b}. \tag{9}
\]
Substituting these values in the third equations (8), cancelling the factor \(\beta\), multiplying \((bc)\), and simplifying we obtain the equality (7). \(\square\)
In accordance with Lemma 1.1, the system (5) has the first integral (6) in the two cases:

1. $3b + 2c = 0$, then in accordance with equalities (9) the integral (6) has the form

$$I = \left(-\frac{3}{2}a y^4 + 2c x^3 y^2 + dx^6\right)\frac{\beta}{2c}$$ (10)

and Hamiltonian function $H = -Ic/(3\beta)$;

2. $ad - bc = 0$, if $3b + 2c \neq 0$, then the integral $c_0 I$ is not a Hamiltonian function for any constant $c_0$; if $3b + 2c = 0$, then the integral (10) and Hamiltonian are proportional to the square $(c_1 y^2 + c_2 x^3)^2$, where $c_1, c_2 = \text{const.}$

Multiplying $x$ and $y$ in the system (5) by the constants, we can reduce 2 from 4 parameters $a, b, c, d$. For example it is possible to take $a = d = 1$.

In [Algaba et al.:2009], systems (3), (5) were studied in the case 1 above. We study them in the case 2.
So

We consider the system

\[
\begin{align*}
\frac{dx}{dt} &= -y^3 - bx^3y + a_0 x^5 + a_1 x^2 y^2, \\
\frac{dy}{dt} &= \left(\frac{1}{b}\right) x^2 y^2 + x^5 + b_0 x^4 y + b_1 x y^3,
\end{align*}
\]

(18)

with arbitrary complex parameters \(a_i, b_i\) and \(b \neq 0\).

\[
\frac{dx}{dt} = x \left( -x^{-1} y^3 - bx^2 y + a_0 x^4 + a_1 xy^2 \right)
\]

\[
\frac{dy}{dt} = y \left( \left(\frac{1}{b}\right)x^2 y + x^5 y^{-1} + b_0 x^4 + b_1 xy^2 \right)
\]
After the power transformation
\[ x = u v^2, \quad y = u v^3 \]  
and time rescaling
\[ dt = u^2 v^7 d\tau, \]
we obtain the system (18) in the form
\[
\begin{align*}
du/d\tau &= -3u - [3b + (2/b)]u^2 - 2u^3 + (3a_1 - 2b_1)u^2v + (3a_0 - 2b_0)u^3v, \\
dv/d\tau &= v + [b + (1/b)]uv + u^2v + (b_1 - a_1)uv^2 + (b_0 - a_0)u^2v^2.
\end{align*}
\]

Under the power transformation (19), the point \( x = y = 0 \) blows up into two straight lines \( u = 0 \) and \( v = 0 \). Along the line \( u = 0 \) the system (20) has a single stationary point \( u = v = 0 \). Along the second line \( v = 0 \) this system has three elementary stationary points
\[
u = 0, \quad u = -\frac{1}{b}, \quad u = -\frac{3b}{2}.
\]
Lemma 3.1 Near the point $u = v = 0$, the system (20) is locally integrable.

Proof. In accordance with Chapter 2 of the book [Bruno:1979], the support of the system (20) consists of the five points $Q = (q_1, q_2)$

\[(0, 0), \quad (1, 0), \quad (2, 0), \quad (1, 1), \quad (2, 1).\] \quad (22)

At the point $u = v = 0$ eigenvalues of the system (20) are $\Lambda = (\lambda_1, \lambda_2) = (-3, 1)$. Only for the first point from (22) $Q = 0$, the scalar product $\langle Q, \Lambda \rangle$ is zero, for the remaining four points (22) it is negative, so these four points lie on the same side of the straight line $\langle Q, \Lambda \rangle = 0$. In accordance with the remark at the end of Subsection 2.1 of Chapter 2 of the book, in such case the normal form consists only of the terms of a right side of the system (20) such that their support $Q$ lies on the straight line $\langle Q, \Lambda \rangle = 0$. But only linear terms of the system (20) satisfy this condition. Therefore at the point $u = v = 0$ the normal form of the system is linear

\[
dz_1/d\tau = -3z_1, \quad dz_2/d\tau = z_2.
\]
(0, 0), (1, 0), (2, 0), (1, 1), (2, 1).

At the point \( u = v = 0 \) eigenvalues of the system (20) are \( \lambda = (\lambda_1, \lambda_2) = (-3, 1) \). Only for the first point from (22) \( Q = 0 \), the scalar product \( \langle Q, \lambda \rangle \) is zero, for the remaining four points (22) it is negative, so these four points lie on the same side of the straight line \( \langle Q, \lambda \rangle = 0 \). In accordance with the remark at the end of Subsection 2.1 of Chapter 2 of the book, in such case the normal form consists only of the terms of a right side of the system (20) such that their support \( Q \) lies on the straight line \( \langle Q, \lambda \rangle = 0 \). But only linear terms of the system (20) satisfy this condition. Therefore at the point \( u = v = 0 \) the normal form of the system is linear

\[
\frac{dz_1}{d\tau} = -3z_1, \quad \frac{dz_2}{d\tau} = z_2.
\]
It is obvious that this normal form satisfies the condition A. So the normalizing transformation converges, and at the point $u = v = 0$ the system (20) has the analytic first integral
\[ z_1 z_2^3 = \text{const.} \]

The proof of local integrability at the point $u = \infty, v = 0$ is similar.
Thus if we must find conditions of local integrability at two other stationary points (21). We will have the conditions of local integrability of the system (18) near the point $X = 0$. 
Let us consider the stationary point $u = -1/b, v = 0$. Below we restrict ourselves to the case $b^2 \neq 2/3$ when a linear part of the system (20), after the shift $u = \tilde{u} - 1/b$, has non-vanish eigenvalues. At $b^2 = 2/3$ the shifted system in new variables $\tilde{u}$ and $v$ has Jordan cell with both zero eigenvalues as the linear part. This case can be studied by using one more power transformation.
To simplify eigenvalues, we change the time at this point once more with the factor $d\tau = (2 - 3b^2)/b^2 \, d\tau_1$. After that we obtain the vector of eigenvalues of the system (20) at this point as $(\lambda_1, \lambda_2) = (-1, 0)$. So the normal form of the system will become
\begin{align}
\frac{dz_1}{d\tau_1} &= -z_1 + z_1 \, g_1(z_2), \\
\frac{dz_2}{d\tau_1} &= z_2 \, g_2(z_2),
\end{align}

where $g_{1,2}(x)$ are formal power series in $x$. Coefficients of these series are rational functions of the parameters of the system $a_0, a_1, b_0, b_1$ and $b$. It can be proved that denominator of each of these rational functions is proportional to some integer degree $k(n)$ of the polynomial $(2 - 3b^2)$. Their numerators are polynomials in parameters of the system
\[ g_{1,2}(x) = \sum_{n=1}^{\infty} \frac{p_{1,2;n}(b, a_0, a_1, b_0, b_1)}{(2 - 3b^2)^{k(n)}} \, x^n. \]
The condition A of integrability for the equation (23) is $g_2(x) = 0$. It is equivalent to the infinite polynomial system of equations

$$p_{2,n}(b, a_0, a_1, b_0, b_1) = 0, \quad n = 1, 2, \ldots.$$  

(24)

According to the Hilbert's theorem on bases in polynomial ideals [Siegel:1956], this system has a finite basis.

We computed the first three polynomials $p_{2,1}$, $p_{2,2}$, $p_{2,3}$ by our program [Edneral:2007]. There are 2 solutions of a corresponding subset of equations (24) at $b \neq 0$

$$a_0 = 0, \quad a_1 = -b_0 b, \quad b_1 = 0, \quad b^2 \neq 2/3$$  

(25)

and

$$a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3.$$  

(26)

Addition of the forth equation $p_{2,4} = 0$ to the subset of equations does not change these solutions.
A calculation of polynomials $p_{2,n}(b, a_0, a_1, b_0, b_1)$ in generic case is technically a very difficult problem. But we can verify some of these equations from the set (24) on solutions (25) and (26) for several fixed values of the parameter $b$. We verified solutions of subset of equations

$$p_{2,n}(b, a_0 = a_1 b, a_1, b_0 = b_1 b, b_1) = 0, \quad n = 1, 2, \ldots, 28.$$ 

at $b = 1$ and $b = 2$. All equations are satisfied, so we can assume that (25) and (26) satisfy the condition A at the stationary point $u = -1/b, v = 0$. 


Let us consider the stationary point $u = -3b/2, v = 0$. We rescale time at this point with the factor $d\tau = (2 - 3b^2)d\tau_2$. After that we get the vector of eigenvalues of the system (20) at this point as $(-1/4, 3/2)$. So the normal form has a resonance of the seventh order

$$
\begin{align*}
&dz_1/d\tau_2 = -(1/4)z_1 + z_1 r_1(z_1^6 z_2), \\
&dz_2/d\tau_2 = (3/2)z_2 + z_2 r_2(z_1^6 z_2),
\end{align*}
$$

(27)

where $r_{1,2}(x)$ are also formal power series, and in (27) they depend on single "resonant" variable $z_1^6 z_2$. Coefficients of these series are rational functions of system parameters $a_0, a_1, b_0, b_1$ and $b$ again. The denominator of each of these functions is proportional to some integer degree $l(n)$ of the polynomial $(2 - 3b^2)$. Their numerators are polynomials in parameters of the system

$$
\begin{align*}
r_{1,2}(x) = \sum_{n=1}^{\infty} \frac{q_{1,2;n}(b, a_0, a_1, b_0, b_1)}{(2 - 3b^2)^{l(n)}} x^n.
\end{align*}
$$
The condition A for the equation (27) is $6 r_1(x) + r_2(x) = 0$. It is equivalent to the infinite polynomial system of equations

$$6 q_{1,n}(b, a_0, a_1, b_0, b_1) + q_{2,n}(b, a_0, a_1, b_0, b_1) = 0, \quad n = 7, 14, \ldots \quad (28)$$

We computed polynomials $q_{1,7}, q_{2,7}$ and solved the lowest equation from the set (28) for the parameters of the solution (26). We have found 5 different two-parameter ($b$ and $a_1$) solutions. With the (26) they are

1) $\quad b_1 = -2a_1, \quad a_0 = a_1b, \quad b_0 = b_1b, \quad b^2 \neq 2/3,$

2) $\quad b_1 = (3/2)a_1, \quad a_0 = a_1b, \quad b_0 = b_1b, \quad b^2 \neq 2/3,$

3) $\quad b_1 = (8/3)a_1, \quad a_0 = a_1b, \quad b_0 = b_1b, \quad b^2 \neq 2/3$ \hfill (29)

and

4) $\quad b_1 = \frac{197-7\sqrt{745}}{24}a_1, \quad a_0 = a_1b, \quad b_0 = b_1b, \quad b^2 \neq 2/3,$

5) $\quad b_1 = \frac{197+7\sqrt{745}}{24}a_1, \quad a_0 = a_1b, \quad b_0 = b_1b \quad b^2 \neq 2/3.$ \hfill (30)
We verified (28) up to \( n = 49 \) for solutions (29) for \( b = 1 \) and \( b = 2 \) and arbitrary \( a_1 \). They are correct. We verified the solution (25) in the same way. It is also correct.

Solutions (30) starting from the order \( n = 14 \) are correct only for the additional condition \( a_1 = 0 \). But for this condition, solutions (30) are a special case of solutions (29). So in accordance with the main supposition we can formulate the theorem.

So,

**Theorem 3.2** Equalities (25), and (29) form a full set of necessary conditions of a local integrability of the system (20) in all its stationary points and a local integrability of system (18) at stationary point \( x = y = 0 \).
About sufficient conditions of an integrability

The conditions resulted in the theorem 3.2 are necessary for local integrability of the system (18) in the zero stationary point. They can be considered from the point of view "experimental mathematics" as sufficient conditions of local integrability, especially considering high enough orders of the checks above. However it is necessary to prove the sufficiency of these conditions by independent methods. It is necessary to do it for each of four conditions (25), (29) in each of stationary points \( u = -3b/2; \ v = 0 \) and \( u = -1/b; \ v = 0 \).

\[
\begin{align*}
a_0 &= 0, \quad a_1 = -b_0 b, \quad b_1 = 0, \quad b^2 \neq 2/3 \\
1) \quad b_1 &= -2a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3, \\
2) \quad b_1 &= (3/2)a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3, \\
3) \quad b_1 &= (8/3)a_1, \quad a_0 = a_1 b, \quad b_0 = b_1 b, \quad b^2 \neq 2/3
\end{align*}
\]
As to a condition (25) it is possible to construct the 1st integral for the both equations (20) and (18) by a method of uncertain coefficients. To within constant multiplier and constant item it will have in variables u; v and x; y forms

At: \( a_0 = 0; \quad a_1 = -b_0 \ b; \quad b_1 = 0 : \)

\[
I_{1uv} = u^2 (3 \ b + 2 \ u) \ v^6;
\]

\[
I_{1xy} = 2 \ x^3 + 3 \ b \ y^2:
\]

There are no the restriction \( b^2 \neq 2/3. \)
For the first condition from (29) the sufficiency is proved by the same way. A calculation of the first integrals gives

At \( b_1 = -2a_1, a_0 = a_1 b, b_0 = b_1 b \):

\[
\begin{align*}
I_{2uv} &= u^2 v^6 (3 b + u (2 - 6 a_1 b v)), \\
I_{2xy} &= 2 x^3 - 6 a_1 b x^2 y + 3 b y^2.
\end{align*}
\] (32)

There are no the restriction \( b^2 \neq 2/3 \).
The second condition from (29) allows to split variables in system (20) and
to solve system in implicit form. We can receive the first integrals in a kind

At $b_1 = 3a_1/2$, $a_0 = a_1b$, $b_0 = b_1b$:

$$
I_{3uv} = \frac{4 - 4a_1 uv + 3^{5/6} a_1}{u^{1/3} v (3b + 2u)^{1/6}} 2F_1(2/3, 1/6; 5/3; -2u/(3b)) u v (3 + 2u/b)^{1/6},
$$

$$
I_{3xy} = \frac{a_1 x^2 (-4 + 3^{5/6})}{y^{4/3}(3b + 2x^3/y^2)^{1/6}} 2F_1(2/3, 1/6; 5/3; -2x^3/(3by^2)) (3 + 2x^3/(by^2))^{1/6} + 4y,
$$

(33)

Where $2F_1$ - a hypergeometric series [8].

Let’s notice, that cubes from $I_{3uv}$ and $I_{3xy}$ are analytical integrals. Convergence
radiuses hypergeometric series in arguments are also are enough for analyticity.

As the local integrals corresponding fourth condition $b_1 = 8a_1/3$, $a_0 = a_1b$, $b_0 = b_1b$, $b^2 \neq 2/3$, we calculated only truncated formal power series on $u$ and $v$ in each of stationary points $u = -3b/2, v = 0$ and $u = -1/b, v = 0$ separately. Attempts to summate of these truncated formal power series by the
2-dimensional Pade approximation had no success.
\begin{equation}
\text{NewEqs} = \text{ExpandAll}[\text{Simplify}[\text{NewEqs1} //. \{a0 \rightarrow b\ a1,\ b0 \rightarrow b\ b1,\ b1 \rightarrow 3\ a1/2\}]]
\end{equation}

\begin{align*}
3\ u[t] + \frac{2\ u[t]^2}{b} + 3\ b\ u[t]^2 + 2\ u[t]^3 + u'[t] &= 0, \\
v'[t] &= v[t] + \frac{u[t]\ v[t]}{b} + b\ u[t]\ v[t] + u[t]^2\ v[t] + \frac{1}{2}\ a1\ u[t]\ v[t]^2 + \frac{1}{2}\ a1\ b\ u[t]^2\ v[t]^2
\end{align*}
The forth case

As the local integrals corresponding fourth condition $b_1 = 8a_1/3, a_0 = \nu_1 b, b_0 = b_1 b, b^2 \neq 2/3$, we calculated only truncated formal power series on $u$ and $v$ in each of stationary points $u = -3b/2, v = 0$ and $u = -1/b, v = 0$.
\[
\text{NewEqs} = \text{ExpandAll} [\text{NewEqs1} /\ {u[t] \rightarrow u[t] - 3 b/2}]
\]

\[
\{ u'[t] = 3 u[t] - \frac{9}{2} b^2 u[t] - \frac{2 u[t]^2}{b} + 6 b u[t]^2 - 2 u[t]^3 - \frac{21}{4} a_1 b^2 v[t] + \frac{63}{8} a_1 b^4 v[t] + 7 a_1 b u[t] v[t] - \frac{63}{4} a_1 b^3 u[t] v[t] - \frac{7}{3} a_1 u[t]^2 v[t] + \frac{21}{2} a_1 b^2 u[t]^2 v[t] - \frac{7}{3} a_1 b u[t]^3 v[t], v'[t] = -\frac{v[t]}{2} + \frac{3}{4} b^2 v[t] + \frac{u[t] v[t]}{b} - 2 b u[t] v[t] + u[t]^2 v[t] - \frac{5}{2} a_1 b v[t]^2 + \frac{15}{4} a_1 b^3 v[t]^2 + \frac{5}{3} a_1 u[t] v[t]^2 - 5 a_1 b^2 u[t] v[t]^2 + \frac{5}{3} a_1 b u[t]^2 v[t]^2 \}
\]

**Integral up to 14 level**

\[
\text{CC} = \frac{1}{1778136624000 b^5 (-2 + 3 b^2)^7} \{ v[t]^6 \left( 197570736000 b^3 (-2 + 3 b^2)^7 (3 b - 2 u[t])^2 u[t] - 158400 a_1 (-2 + 3 b^2)^7 \\
(16838415 b^7 - 2 u[t] (-50515245 b^6 - 2 u[t]) (48270123 b^5 + 2 u[t] (-12030679 b^4 + 256 u[t]) (4669 b^3 + 406 b^2 u[t] + 84 b u[t]^2 + 24 u[t]^3)))) \right) \\
(649274876775 b^8 + 8 u[t] (63144056250 b^6 - 33545193245 b^5 u[t] + 30239668800 b^4 u[t]^2 - 93682328298 b u[t]^3 + 5535236096 u[t]^4) v[t]^3 + \\
2016 a_1 b^6 (-2 + 3 b^2)^7 (6429874876775 b^8 + 2 u[t] (39398919615 b^7 - 4 u[t] (325815994845 b^6 - 38024606060 b u[t] + 16145233018 u[t]^2)))) v[t]^4 - \\
2016 a_1 b^8 (-2 + 3 b^2)^7 (2997829195515 b^9 + 2 u[t] (-738061851285 b^8 - 4824647071236 b u[t] + 7317594385688 u[t]^2)) v[t]^5 + \\
108 a_1 b^{10} (232370921807640 b^2 (-2 + 3 b^2)^7 - 231199232071584 b (-2 + 3 b^2)^7 u[t] + \\
(722246971616139704 - 3 b^2 (252623675681399684 - 1135498901584660798 b^2 + 283300088479704475 b^4 - \\
4240834751647328100 b^6 + 3842589615793018812 b^8 - 2035396723453784766 b^{10} + 555708882541760847 b^{12}) u[t]^2)) v[t]^6 - \\
972 a_1 b^{12} (98415926253904 b (-2 + 3 b^2)^7 + (17777857387692136 - 168504738170678148 b^2 + 837963682786413886 b^4 - 2089162837802287195 b^6 + \\
3125077680973902180 b^8 - 283840825234893548 b^{10} + 1533306041731743102 b^{12} - 448118022172751919 b^{14} u[t] v[t]) v[t]^7 + \\
243 a_1 b^{14} (-180615143960489672 + 3 b^2 (632315768261801332 - 284672859615971874 b^2 + 7122567859563127205 b^4 - \\
10692518365074219420 b^6 + 9597428189094373956 b^8 - 4684612178989911618 b^{10} + 88429302512475521 b^{12}) v[t]^8))
\]

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Derivation along the equation

\[ \varphi = \text{Simplify} \left[ \frac{D[CC, t]}{.} \right. \]

\[ \{ u'[t] \to 3u[t] - \frac{9}{2} b^2 u[t] - \frac{2 u[t]^2}{b} + 6 b u[t]^2 - 2 u[t]^3 - \frac{21}{4} a b^2 v[t] + \frac{63}{8} a b^4 v[t] + 7 a b u[t] v[t] - \frac{63}{4} a b^3 u[t] v[t] - \]

\[ \frac{7}{3} a u[t]^2 v[t] + \frac{21}{2} a b^2 u[t]^2 v[t] - \frac{7}{3} a b u[t]^3 v[t], \]

\[ v'[t] \to -\frac{v[t]}{2} + \frac{3}{4} b^2 v[t] + \frac{u[t] v[t]}{b} - 2 b u[t] v[t] + u[t]^2 v[t] - \frac{5}{2} a b v[t]^2 + \frac{15}{4} a b^3 v[t]^2 + \frac{5}{3} a u[t] v[t]^2 - \]

\[ 5 a b^2 u[t] v[t]^2 + \frac{5}{3} a b u[t]^2 v[t]^2 \} ] \]
Expand $[q q/.v[t]->u[t]]$
\[
\text{NewEqs} = \text{ExpandAll}[\text{NewEqs1} /. u[t] \to u[t] - 1/b]
\]

\[
\begin{align*}
\{ u'[t] &= -2 u[t] + 3 b^2 u[t] + 4 b u[t]^2 - 3 b^3 u[t]^2 - 2 b^2 u[t]^3 - \frac{7}{3} a_1 b u[t] v[t] + \frac{14}{3} a_1 b^2 u[t]^2 v[t] - \frac{7}{3} a_1 b^3 u[t]^3 v[t], \\
\{ v'[t] &= -b u[t] v[t] + b^3 u[t] v[t] + b^2 u[t]^2 v[t] - \frac{5}{3} a_1 b^2 u[t] v[t]^2 + \frac{5}{3} a_1 b^3 u[t]^2 v[t]^2 \}
\end{align*}
\]

**Integral up to 7 level**

\[
CC = \frac{1}{58320} (v[t], (58320 (-2 + 3 b^2)^7 - 4455 b^7 (-27 + b^2 (189 + 2 b^2 (-567 + b^2 (1575 + 8 b^2 (-315 + 297 b^2 - 154 b^4 + 34 b^5)))))) u[t]^2 - \\
81 b^6 u[t] (5 (-2 + 3 b^2)^5 (189 + 2 b^2 (-567 + 2835 b^2 - 6300 b^4 + 7560 b^6 - 4752 b^8 + 1232 b^{10})) + a_1 b (945 + 2 b^2 (-3255 + 35421 b^2 - 55354 b^4 - 55334 b^6 - 30616 b^8 + 7200 b^{10})) v[t]) - \\
54 b^5 u[t] (45 (-2 + 3 b^2)^2 (-21 + 105 b^2 - 420 b^4 + 700 b^6 - 560 b^8 + 176 b^{10}) + \\
a_1 b v[t] (3 (-2 + 3 b^2)^2 (-315 + 2 b^2 (945 - 8136 b^2 + 9538 b^4 - 6356 b^6 + 1760 b^8)) + 5 a_1 b (-423 + 2 b^2 (-2223 - 16438 b^2 + 6168 b^4 - 3952 b^6 + 952 b^8)) v[t])) - \\
60 b^4 u[t] v[t] (729 (-1 + b) (1 + b) (2 - 3 b^2)^6 - a_1 b (-3 + 8 b^2) v[t] (-4 - 6 b^2 + 7 a_1 b v[t]) (9 (2 - 3 b^2)^2 - 7 a_1 b v[t] (6 - 9 b^2 + 7 a_1 b v[t]) (9 (2 - 3 b^2)^2 + 7 a_1 b v[t] (-6 + 9 b^2 + 7 a_1 b v[t]))) - \\
90 b^3 u[t] v[t]^2 (135 (-2 + 3 b^2)^3 (3 + 4 (-1 + b) b^2 (1 + b) (3 - 6 b^2 + 4 b^4)) + a_1 b v[t] (9 (2 - 3 b^2)^2 (45 + 2 b^2 (-115 + b^2 (737 + 192 b^2 (-3 + b^2)))) + \\
2 a_1 b v[t] (3 (-2 + 3 b^2) (153 - 1350 b^2 + 7708 b^4 - 2308 b^6 + 696 b^8) + a_1 b (1182 - 16041 b^2 + 77322 b^4 + 4720 b^6 + 2560 b^8)) v[t])) - \\
120 b^2 u[t] v[t]^3 (81 (2 - 3 b^2)^4 (-3 + 9 b^2 - 18 b^4 + 10 b^6) + a_1 b v[t] (27 (-2 + 3 b^2)^3 (-9 + 38 b^2 - 164 b^4 + 64 b^6) + \\
a_1 b v[t] (9 (2 - 3 b^2)^2 (-63 + 444 b^4 - 1808 b^6 + 376 b^8) + a_1 b v[t] (3 (-2 + 3 b^2) (-489 + 5256 b^2 - 20246 b^4 + 4544 b^6) + 7 a_1 b (-684 + 9329 b^2 - 34150 b^4 + 12392 b^6)) v[t])))) - \\
360 b^2 u[t]^2 v[t] (81 (-2 + 3 b^2)^5 (1 - 2 b^2 + 2 b^4) + a_1 b v[t] (27 (2 - 3 b^2)^4 (3 - 10 b^2 + 24 b^4) + a_1 b v[t] (54 (-2 + 3 b^2)^3 (4 - 19 b^2 + 44 b^4) + \\
7 a_1 b v[t] (6 (2 - 3 b^2)^2 (9 - 55 b^2 + 148 b^4) + 7 a_1 b v[t] (-6 - 125 b^2 - 702 b^4 + 792 b^6 + 49 a_1 b (-1 + 2 b^2 - 8 b^4)) v[t])))))
\]

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Derivation along the equation

\[ q_1 = \text{Simplify}\left[ D[CC, t] / . \{ u'[t] \rightarrow -2 u[t] + 3 b^2 u[t] + 4 b u[t]^2 - 3 b^3 u[t]^2 - 2 b^2 u[t]^3 - \frac{7}{3} a_1 b u[t] v[t] + \frac{14}{3} a_1 b^2 u[t]^2 v[t] - \frac{7}{3} a_1 b^3 u[t]^3 v[t], v'[t] \rightarrow -b u[t] v[t] + b^3 u[t] v[t] + b^2 u[t]^2 v[t] - \frac{5}{3} a_1 b^2 u[t] v[t]^2 + \frac{5}{3} a_1 b^3 u[t]^2 v[t] \} \right] \]
Expand[$qq/\cdot v[t]\rightarrow u[t]]$

\[
\begin{align*}
429b^0u[t]^9 & + 231a_1b^0u[t]^7 + 35a_1^2b^0u[t]^9 + 5435a_1b^0u[t]^9 + 377a_1^2b^0u[t]^9 + 2842a_1b^0u[t]^9 + 4802a_1^2b^0u[t]^9 + 117649a_1^3b^0u[t]^9 + 429b^{10}u[t]^9 + 2695a_1b^{10}u[t]^9 \\
& + O(u[t]^9)
\end{align*}
\]

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Conclusion

For the planar 5-parametrical system (18) are found all necessary sets of conditions on a parameters at which the systems (18) is local integrable near degenerate points $X = 0$. They are 4 sets of conditions on the system parameters. For 3 from these sets by the independent methods it is shown, that they are also sufficient for the local integrability of the system (18) and at these families of parameters first integrals was calculated, i.e. system at these values of parameters is global integrable. For the fourth set of parameters we received only approximations of local integrals in the form of truncated formal series on system variables. Also the point $b^2 = 2/3$ have been excluded from the analysis of this fourth set of parameters.
References


Thanks for attention!